Journal of Applied Mathematics and Physics (ZAMP)
Vol. 40, March 1989
0044-2275/89/020216-14 \$ 1.50 + 0.20/0


# Disturbed critical surface waves in a channel of arbitrary cross section

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## 1. Introduction

We consider the flow of an inviscid fluid of constant density in a channel with arbitrary cross section. Let $L$ and $H$ be longitudinal and transverse scales respectively. A small number $\varepsilon$ is defined by $\varepsilon = (H/L)^2$. When cross section of the channel varies at order $O(\varepsilon^3)$ in the longitudinal direction and the ambient shear flow is $U_0(y, z)$, the governing equation of the first order elevation of free surface, derived first by Peters [1], is a Korteweg-de Vries (K-dV) equation. This K-dV equation possesses a traveling soliton solution. Shen [2] considered $O(\varepsilon)$ order variation of the cross section of the channel in the longitudinal direction and zero ambient shear flow, and derived a governing equation for the first order elevation of the free surface, which is an equation of K-dV type with variable coefficients. This equation has been successfully used to study soliton fission in channels (Zhong and Shen [3]). Here we consider $O(\varepsilon^2)$ order variation of the cross section of the channel in the longitudinal direction and a nonzero ambient shear flow $U_0(y, z)$ which is near its critical state $U_c(y, z)$. A forced K-dV (fK-dV) equation with constant coefficients is derived. The forcing term is due to the $O(\varepsilon^2)$ variation of the cross section and pressure disturbance on the free surface. The derived fK-dV equation is the same as the one obtained by Mei [4] for a channel of rectangular cross section and constant shear flow. For channels with different cross sections, the critical states, $U_c(y, z)$, are different. If $U_0(y, z) \equiv F$ is constant, the critical state $U_c(y, z) = F_c$ is also constant. The surface waves correspond to $F > F_c (F < F_c)$ are referred as supercritical (subcritical) waves. When the forcing function in the fK-dV equation is of compact support and far upstream free surface elevation is prescribed as zero, supercritical solutions of steady state die out far downstream, i.e. the steady supercritical solutions are solitary waves. For the same forcing function and the same upstream data, subcritical solutions of steady state oscillate far downstream, which are cnoidal waves. It is further shown in this paper that there do not exist solitary wave solutions of steady state when the shear flow takes its critical speed $F_c$. This gives a partial proof of the conjecture due to Wu and Wu [5] that "The transcritical motion does not approach a steady state".

The aforementioned results agree qualitatively well with those obtained by Shen, et al. [6], Vanden-Broeck [7] and Forbes and Schwartz [8], all of whom considered two dimensional channel flows.

In section 2, the forced K-dV (fK-dV) equation is derived. The existence and nonexistence of supercritical solutions of steady state is discussed in section 3. Numerical methods and results are described in section 4.

## 2. Derivation of a forced *K-dV* equation

Let the $x^*$-axis be aligned along the longitudinal direction of the channel, the $y^*$-axis along the spanwise direction and $z^*$-axis vertically in the opposite direction to gravitation. The $x^*$–$y^*$ plane is placed on the undisturbed free surface. The equation of the boundary of the channel is $h^*(x^*, y^*, z^*) = 0$. The equation of free surface is denoted by $z^* = \eta^*(x^*, y^*, t^*)$ where $t^*$ stands for the time coordinate and the superscript * characterizes dimensional quantities. Then the equations of motion and boundary conditions are

$$u^*_{x^*} + v^*_{y^*} + w^*_{z^*} = 0, \tag{1}$$

$$u^*_{t^*} + u^*u^*_{x^*} + v^*u^*_{y^*} + w^*u^*_{z^*} = -\frac{1}{\rho^*}p^*_{x^*}. \tag{2}$$

$$v^*_{t^*} + u^*v^*_{x^*} + v^*v^*_{y^*} + w^*v^*_{z^*} = -\frac{1}{\rho^*}p^*_{y^*}, \tag{3}$$

$$w^*_{t^*} + u^*w^*_{x^*} + v^*w^*_{y^*} + w^*w^*_{z^*} = -g - \frac{1}{\rho^*}p^*_{z^*}; \tag{4}$$

on the free surface

$$z^* = \eta^*(x^*, y^*, t^*),$$

$$\eta^*_{t^*} + u^*\eta^*_{x^*} + v^*\eta^*_{y^*} - w^* = 0, \tag{5}$$

$$p^* = \bar{p}^*(x^*); \tag{6}$$

on the wall of the channel

$$h^*(x^*, y^*, z^*) = 0,$$

$$u^*h^*_{x^*} + v^*h^*_{y^*} + w^*h^*_{z^*} = 0. \tag{7}$$

Here $(u^*, v^*, w^*)$ is velocity; $\rho^*$ is density; $p^*$ is pressure; $g$ is the gravitational acceleration constant; and $\bar{p}^*$, which is assumed to be function of only $x^*$, is the disturbance pressure on the free surface (see Fig. 1).

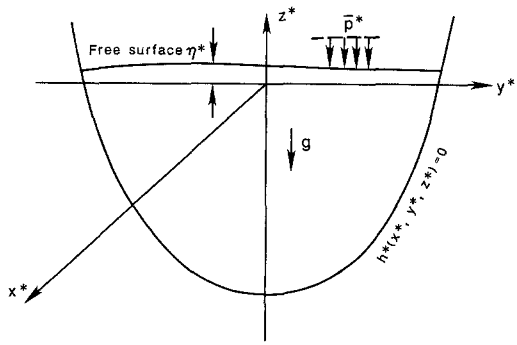


Figure 1
An ideal fluid flow through a channel of arbitrary cross section with the free surface of the flow disturbed by a distributed pressure $\bar{p}^*(x^*)$.

To nondimensionalize (1–7), the following dimensionless quantities are introduced.

$$\varepsilon = \left(\frac{H}{L}\right)^2 \ll 1, \quad t = \varepsilon^{3/2}\sqrt{\frac{g}{H}}\,t^*,$$

$$(x, y, z) = \frac{1}{H}(\varepsilon^{1/2}x^*, y^*, z^*),$$

$$\eta = \frac{\eta^*}{H}, \quad p = \frac{p^*}{\rho^* g H}, \quad \bar{p} = \varepsilon^2 \frac{\bar{p}^*}{\rho g H},$$

$$(u, v, w) = \frac{1}{\sqrt{gH}}(u^*, \varepsilon^{-1/2}v^*, \varepsilon^{-1/2}w^*),$$

$$h_1 = \varepsilon^{-5/2}h^*_{x^*}, \quad h_2 = h^*_{y^*}, \quad h_3 = h^*_{z^*}.$$

In terms of these dimensionless quantities and by approximating the boundary conditions on the free surface around $z = 0$, (1–7) can be written as

$$u_x + v_y + w_z = 0, \tag{8}$$

$$\varepsilon u_t + uu_x + p_x + vu_y + wu_z = 0, \tag{9}$$

$$\varepsilon v_t + \varepsilon uv_x + p_y + vv_y + wv_z = 0, \tag{10}$$

$$\varepsilon w_t + \varepsilon uw_x + 1 + p_x + vw_y + ww_z = 0; \tag{11}$$

on $z = 0$,

$$w - \varepsilon\eta_t - u\eta_x - v\eta_y = 0, \tag{12}$$

$$p = \varepsilon^2\bar{p} + \eta; \tag{13}$$

on $h = 0$,

$$\varepsilon^2 u h_1 + v h_2 + w h_3 = 0. \tag{14}$$

For an ambient shear flow $U_0(y, z) = u_0(y, z) + \varepsilon\lambda + 0(\varepsilon^2)$, we assume asymptotic expansion of the following form:

$$(u, v, w, \eta, p) = (u_0(y, z), 0, 0, 0, -z) + \varepsilon(u_1 + \lambda, v_1, w_1, \eta_1, p_1) + \varepsilon^2(u_2, v_2, w_2, \eta_2, p_2) + 0(\varepsilon^3). \tag{15}$$

Inserting (15) into (8–14) and assembling the resulting equations according to the powers of $\varepsilon$, it follows that the equations of the order $\varepsilon$ and $\varepsilon^2$ are as below.

$O(\varepsilon)$:

$$u_{1x} + v_{1y} + w_{1z} = 0, \tag{16}$$

$$u_0 u_{1x} + p_{1x} + v_1 u_{0y} + w_1 u_{0z} = 0, \tag{17}$$

$$p_{1y} = 0, \tag{18}$$

$$p_{1z} = 0; \tag{19}$$

on $z = 0$,

$$w_1 - u_0 \eta_{1x} = 0, \quad p_1 = \eta_1; \tag{20}$$

on $h = 0$,

$$v_1 h_2 + w_1 h_3 = 0. \tag{21}$$

$O(\varepsilon^2)$:

$$u_{2x} + v_{2y} + w_{2z} = 0, \tag{22}$$

$$u_{1t} + u_0 u_{2x} + (u_1 + \lambda) u_{1x} + p_{2x} + v_1 u_{1y} + v_2 u_{0y} + w_1 u_{1z} + w_2 u_{0z} = 0, \tag{23}$$

$$u_0 v_{1x} + p_{2y} = 0, \tag{24}$$

$$u_0 w_{1x} + p_{2z} = 0, \tag{25}$$

on $z = 0$,

$$w_2 - \eta_{1t} - u_0 \eta_{2x} - (u_1 + \lambda)\eta_{1x} - v_1 \eta_{1y} = 0, \tag{26}$$

$$p_2 = \bar{p} + \eta_2; \tag{27}$$

on $h = 0$,

$$v_2 h_2 + w_2 h_3 = -u_0 h_1. \tag{28}$$

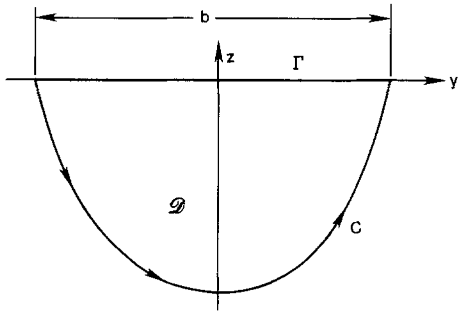


Figure 2
$\mathscr{D}$ is the integration domain of equation (29).

From (16–21), one can derive that (see Fig. 2)

$$\left(\int_{\mathscr{D}}\int \frac{dy\,dz}{u_0^2(y,z)} - \int_{\Gamma} ds\right)\eta_{1x} = 0.$$

For nontrivial solutions, $\eta_{1x} \not\equiv 0$. It follows the dispersion relation,

$$\int_{\mathscr{D}}\int \frac{dy\,dz}{u_0^2(y,z)} = b. \tag{29}$$

Any shear flow $U_0(y,z)$ with $U_0(y,z) = u_0(y,z)$ satisfying (29) is called a critical shear flow. By (24–25),

$$\left(\frac{p_{2y}}{u_0^2}\right)_y + \left(\frac{p_{2z}}{u_0^2}\right)_z = -\frac{p_{1xx}}{u_0^2}.$$

Assume

$$p_2 = -\phi(y,z)p_{1xx}(x,t) + C_1(x,t), \tag{30}$$

then $\phi$ satisfies

$$\nabla\left(\frac{\nabla\phi}{u_0^2}\right) = \frac{1}{u_0^2} \quad \text{in} \quad \mathscr{D}, \tag{31}$$

$$\phi_z = u_0^2 \qquad \text{on} \quad \Gamma, \tag{32}$$

$$\phi_n = 0 \qquad \text{on} \quad C \tag{33}$$

where $\nabla = \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and $\phi_n$ is the outward normal derivation of $\phi$ on $C$.

By (16),

$$(u_1, v_1, w_1) = \left(-\nabla\left(\frac{\nabla\phi}{u_0}\right)p_1, \frac{\nabla\phi}{u_0}p_{1x}\right). \tag{34}$$

Multiplying (22) by $u_0^{-1}$ and (23) by $-u_0^{-2}$, and integrating the sum of the resulting equations over $\mathscr{D}$, it follows from (29), (30) and (34) that

$$m_1\eta_{1t} + \lambda m_1\eta_{1x} + m_2\eta_1\eta_{1x} + m_3\eta_{1xxx} = -f(x) \tag{35}$$

where

$$m_1 = -2 \int\int_{\mathscr{D}} \frac{dy\,dz}{u_0^3}, \tag{36}$$

$$m_2 = 3 \int\int_{\mathscr{D}} \frac{dy\,dz}{u_0^4} - \int_{\Gamma} \left(\frac{\phi_y}{u_0^2}\right)_y ds, \tag{37}$$

$$m_3 = \int\int_{\mathscr{D}} \left|\frac{\nabla\phi}{u_0}\right|^2 dy\,dz, \tag{38}$$

$$f(x) = b\bar{p}_x + h_1(x) \int_C \frac{ds}{\sqrt{h_2^2 + h_3^2}}. \tag{39}$$

Equation (35) is the forced K-dV equation we desired to derive. To determine $m_1$, $m_2$ and $m_3$, one needs to solve the Neumann problem (31–33). It seems that it is rather difficult to find an analytical solution of (31–33) when $\mathscr{D}$ is neither a rectangular nor triangular region and $u_0(y, z)$ not a constant.

When $u_0$ is constant and $\mathscr{D}$ is a rectangle (see Fig. 3), $\phi_z = z + d, \phi_y = 0$, $m_1 = -\dfrac{2b}{\sqrt{d}}$, $m_2 = 3\dfrac{b}{d}$, $m_3 = \dfrac{bd^3}{3}$, $f(x) = (\bar{p}_x + h_1(x))b$. This agrees with Mei's equation (2.44) [4].

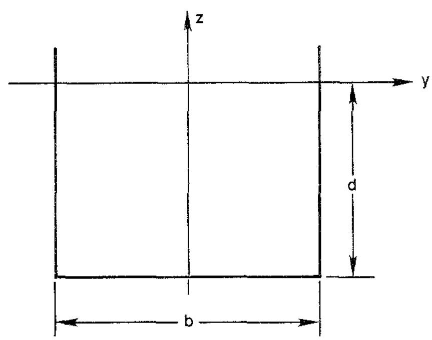


Figure 3
Cross section of a rectangular channel.

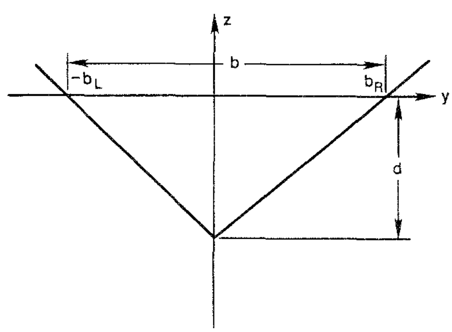


Figure 4
Cross section of a triangular channel.

When $u_0$ is constant and $\mathscr{D}$ is a triangle (see Fig. 4), $\phi = \frac{1}{4}(y^2 + (z+d)^2)$, $m_1 = -2\sqrt{2}\frac{b}{\sqrt{d}}$, $m_2 = 5\frac{b}{d}$, $m_3 = \frac{1}{4d}[bd^2 + \frac{1}{3}(b_L^3 + b_R^3]$, $f(x) = (\bar{p}_x + h_1(x))b$.

## 3. Steady supercritical solutions of (35)

Assume the shear flow is along the positive $x$-axis. Then $u_0(y, z) \geq 0$ and satisfies (29). Thus waves corresponding to $\lambda > 0 (\lambda < 0)$ are supercritical (subcritical) solutions of (35). By equations (36–38), $m_1 < 0$, $m_2 > 0$, $m_3 > 0$. The steady state of (30) can be written as

$$\lambda\eta_{1x} + 2\alpha\eta_{1x} + \beta\eta_{1xxx} = r'(x) \tag{40}$$

where $\alpha = \frac{m_2}{2m_1} < 0$, $\beta = \frac{m_3}{m_1} < 0$, $r'(x) = -\frac{f(x)}{m_1}$. If $\eta_1(x)$ is a solitary wave solution of (40), then it yields

$$\lambda\eta_1 + \alpha\eta_1^2 + \beta\eta_{1xx} = r(x), \tag{41}$$

$$\eta_1(\pm\infty) = 0. \tag{42}$$

In practical applications, it is very often that $r(x) \geq 0$ and $r \in C_0(\mathbb{R})$. Let $x_- = \inf supp(r)$ and $x_+ = \sup supp(r)$.

**Theorem 1.** There exists at least one solution to the problem (41–42) as $\lambda$ is sufficiently large.

**Proof:** We define a complete metric space $B$ as

$$B = \{u \mid u \in C(-\infty, \infty), \; \|u\| = \sup_{-\infty < x < \infty} \exp(v|x|)|u(x)| \leq M$$
$$\text{for some given positive constant } M\}.$$

Here $v = (-\lambda/\beta)^{1/2}$. A contraction mapping theorem in the space $B$ will be used to prove Theorem 1. Equations (41–42) can be converted into the intergral equation

$$\eta_1(x) = \frac{1}{\beta}\int_{-\infty}^{\infty} K(x, \xi)(\alpha\eta_1^2 - r)(\xi)\, d\xi \equiv T(\eta_1) \tag{43}$$

where

$$K(x, \xi) = \frac{1}{2v}\exp(-v|\xi - x|). \tag{44}$$

It is not difficult to show that if

$$\frac{1}{2|\beta|v}\left(\frac{4|\alpha|M}{3v}+\frac{2\max\{\exp(2v|x_-|),\exp(2v|x_+|)\}}{M}\right.$$
$$\left.\cdot\int_{x_-}^{x_+}\cosh(vx)\,|r(x)|\,dx\right)\leq 1, \tag{45}$$

and

$$\frac{4\alpha M}{3v^2\beta}<1, \tag{46}$$

then $T$ is a contraction map in $B$. We notice that (45–46) can always be satisfied as $\lambda$ is sufficiently large. So $T(\eta_1)=\eta_1$ has a unique solution in $B$. This solution approaches zero as $r(x)\equiv 0$ by (45). Also by (45), this solution goes to zero as $\lambda$ goes to infinity. It follows from (43–44) that if $\eta_1(x)$ is continuous, then $T(\eta_1)$ is twice differentiable. Therefore $\eta_1=T(\eta_1)$ is a classical solution of (41–42). The proof is finished.

The results of our numerical computations indicate that (41–42) has actually more than one solution. For $r(x)\geq 0$, this result is stated as follows.

**Theorem 2.** If $\alpha<0$, $\beta<0$, and $r(x)\geq 0$, then there exists $\lambda_c>0$ such that (41–42) has (i) at least two solutions for $\lambda>\lambda_c$; (ii) no solution for $0\leq\lambda<\lambda_c$.

**Proof:** If $r(x)\equiv 0$, then solutions of (41–42) are well known [9], and satisfy the claim of the theorem. So we shall assume that $r(x)\not\equiv 0$. We then claim that every solution of (41–42) is positive. If the claim failed to hold, for any solution $\psi(x)$ of (41–42) there would exist a point $a\in\mathbb{R}$ such that

$$\psi(a)\leq 0,$$
$$\psi''(a)>0.$$

By (41),

$$0>\beta\psi''(a)=r(a)-\lambda\psi(a)-\alpha\psi^2(a)\geq 0.$$

This contradiction shows that $\psi(x)\geq 0$. Since $r(x)\not\equiv 0$, $\psi(x)\equiv 0$ is not a solution of (41–42). Hence $\psi(x)>0$, for real $x$.

If $\eta_1=\psi(x)$ is a positive solution, $\psi$ must be bounded. Define $\Theta=\{x\,|\,x\in\mathbb{R},\psi'(x)=0\}$. By Theorem 1, $\Theta\neq\emptyset$. Let $N=\psi(z)>0$, $z=\inf\Theta$. So $\psi(x)$ is monotically increasing in $(-\infty,z)$. $\psi$ has an inverse function with parameter $N$: $x=x(\psi,N)$.

Multiplying (41) by $\psi'(x)$ and integrating the resulting equation with respect to $x$ from $-\infty$ to $z$, we have

$$\lambda = -\tfrac{2}{3}\alpha N + \frac{2}{N^2}\int_0^N r(x(\psi, N))\, d\psi. \tag{47}$$

By a standard argument, $\psi$ depends on $\lambda$ and $N$ continuously when such an $N$ exists [10]. So the curve $S$ on $N$–$\lambda$-plane defined by (47) is a continuous curve. The number of positive solutions when $\lambda = \lambda_0 > 0$, therefore, is equal to the number of intersections of the horizontal line $\lambda = \lambda_0$ with $S$.

By using the mean value theorem to evaluate $\int_0^N r(x(\psi, N))\, d\psi$, we obtain that $N = 0$ and $\lambda = -\dfrac{2\alpha}{3} N$ are two asymptotes of $S$. The fact that $\alpha < 0$ and $r(x) \geq 0$, $r(x) \not\equiv 0$ implies that $\lambda > 0$. Hence $\lambda = \lambda(N)$ has a minimum $\lambda_c > 0$. Therefore, $\lambda = \lambda_0 > \lambda_c$ intersects with $S$ at least two points and $\lambda = \lambda_1 < \lambda_c$ intersects with $S$ at no points. This completes the proof.

Shen [11] recently showed that if solutions of (41–42) are ordered, then (41–42) has exactly two solutions. Here the solutions of (41–42) $\psi_1(x)$ and $\psi_2(x)$ being ordered means that $\psi_1(x) \neq \psi_2(x)$ for any real $x$.

## 4. Numerical methods and results

Let us first consider the supercritical case, i.e., $\lambda > 0$. The difficulty in finding numerical solutions of (41–42) is to distinguish one solution from the other with the same parameter $\lambda$ and the same boundary data at $-\infty$ and $+\infty$. We resolve this difficulty by solving (41–42) analytically from $-\infty$ to $x_- = \inf supp(r)$,

$$\eta_1(x) = -\frac{3\lambda}{2\alpha}\operatorname{sech}^2\sqrt{\frac{-\lambda}{4\beta}}\,(x - L_0), \quad x \leq x_-, \tag{48}$$

and to determine the phase shift $L_0$.

A new quantity $B(x, L_0)$ is introduced as follows

$$\begin{aligned} B(x, L_0) &= \int_{x_-}^{x} r(t)\eta_1'(t)\, dt \\ &= \frac{\beta}{2}(\eta_1'(x))^2 + \left(\frac{\lambda}{2} + \frac{\alpha}{3}\eta_1(x)\right)\eta_1^2(x). \end{aligned} \tag{49}$$

It is clear that if $B(x_+, L_0) = 0$, $x_+ = \sup supp(r)$, then $B(x, L_0) \equiv 0$ for all $x \geq x_+$. It is well known that [9, 12]

$$\frac{\beta}{2}(\eta_1'(x))^2 + \left(\frac{\lambda}{2} + \frac{\alpha}{3}\eta_1(x)\right)\eta_1^2(x) = 0, \quad x > x_+, \tag{50}$$

$$\eta_1(x_+) = \eta_+ > 0, \quad \eta_+ < -\frac{3\lambda}{2\alpha}, \tag{51}$$

$$\eta_1(\infty) = 0 \tag{52}$$

has a unique solution.

Therefore if we can solve the following initial value problem

$$\lambda\eta_1 + \alpha\eta_1^2 + \beta\eta_{1xx} = 0 \quad \text{for } x > x_-, \tag{53}$$

with initial conditions

$$\eta_1(x_-) = -\frac{3\lambda}{2\alpha}\operatorname{sech}^2\sqrt{\frac{-\lambda}{4\beta}}(x_- - L_0), \tag{54}$$

$$\eta_{1x}(x_-) = -\sqrt{\frac{-\lambda}{\beta}}\,\eta_1(x_-)\tanh\sqrt{\frac{-\lambda}{4\beta}}(x_- - L_0) \tag{55}$$

up to $x_+$ for a given $L_0$ such that $B(x_+, L_0) = 0$ and $0 < \eta_1(x_+) < -\dfrac{3\lambda}{2\alpha}$ then we have a global solution of (41–42). Therefore

$$B(x_+, L_0) = 0, \quad 0 < \eta_1(x_+) < -\frac{3\lambda}{2\alpha} \tag{56}$$

is the condition to determine $L_0$. The number of solutions for $L_0$ of (56) is the number of solutions of (41–42).

We have performed detailed computations for the case of triangular channel as $d = \frac{1}{4}$, $b_L = b_R = \dfrac{b}{2}$, $b = \sqrt{16\sqrt{2} - \frac{3}{4}}$, and

$$r(x) = \begin{cases} R\sqrt{1 - x^2}, & |x| \le 1, \\ 0, & |x| > 1 \end{cases}$$

where $R$ is a positive constant. Namely we solve the following problem

$$\eta_{1xx} = 3\left(\lambda\eta_1 - \frac{5}{2\sqrt{2}}\eta_1^2 - r(x)\right), \quad x \in \mathbb{R}, \tag{57}$$

$$\eta_1(\pm\infty) = 0 \tag{58}$$

numerically.

Based on the above analysis, the numerical procedure is carried out as follows. Given a trial value of $L_0$, we use a subroutine DVERK in IMSL (International Mathematical and Statistical Libraries) to solve the following initial value problem

$$\eta_{1xx} = 3\left(\lambda\eta_{1x} - \frac{5}{2\sqrt{2}}\eta_1^2 - r(x)\right), \quad -1 < x \le 1, \tag{53'}$$

$$\eta_1(-1) = \frac{3\sqrt{2}}{5}\lambda\operatorname{sech}^2\sqrt{\frac{3\lambda}{4}}(-1 - L_0), \tag{54'}$$

$$\eta_{1x}(-1) = -\sqrt{3\lambda}\eta_1(-1)\tanh\sqrt{\frac{3\lambda}{4}}(-1-L_0), \tag{55'}$$

and compute

$$B(1, L_0) = \frac{\beta}{2}(\eta_1'(1))^2 + \left(\frac{\lambda}{2} - \frac{5\eta_1(1)}{6\sqrt{2}}\right)\eta_1^2(1).$$

Using a do loop for $L_0$, a function $B(1, L_0)$ *vs* $L_0$ can be plotted. It turns out that $B(1, L_0)$ has two zeros as were expected. For $\lambda = 2.2$, $R = 1.0$, the two zeros are at $L_{01} = -0.126297$ and $L_{02} = 0.072517$. Once having $L_{01}$ and $L_{02}$, we can solve (53′–55′) from $-1$ to any positive right boundary $R_+$ instead to 1. Thus two solutions of (57–58) are obtained. The numerical results obtained are shown in Figs. 5 and 6. For $R = 1.0$, we find $\lambda_c = 2.075$. For the parameter $\lambda > \lambda_c$, there exist two solutions of the problem (50–51). The upper branch corresponds to the perturbation, due to the disturbance $r(x)$, of solitary waves in a channel of uniform cross section. The lower branch corresponds to the perturbation, also due to the disturbance $r(x)$, of a null solution in a channel of uniform cross section. Thus as $r$ approaches zero, the solution diagram degenerates into its asymptotes. We conjecture that the upper branch corresponds to stable solutions and the lower branch corresponds to unstable solutions. The proof of this conjecture for $r \equiv 0$ was

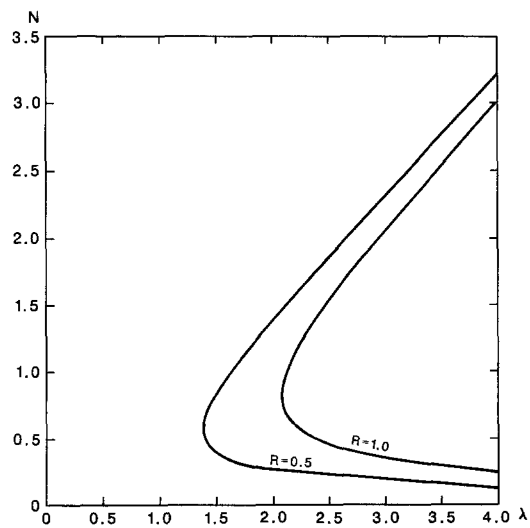


Figure 5
Supercritical solution diagrams of (41–42). The cross section of the channel is a triangle with $d = \frac{1}{4}$, $b_L = b_R = \frac{1}{2}\sqrt{16\sqrt{2} - \frac{3}{4}}$. The disturbance function $r(x)$ is

$$r(x) = \begin{cases} R\sqrt{1-x^2}, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

The right curve corresponds to $R = 1.0$ and at its turning point $\lambda = \lambda_c = 2.075$. The left curve corresponds to $R = 0.5$ and $\lambda_c = 1.379$. Also see equation (47).

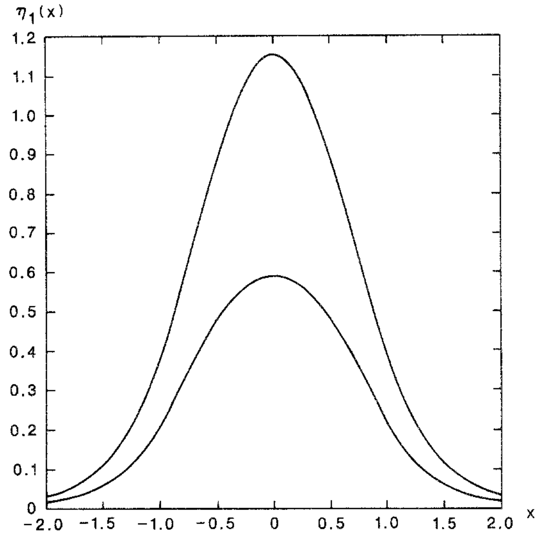


Figure 6
Two supercritical solutions of (50–51) as $\lambda = 2.2$ and $R = 1.0$.

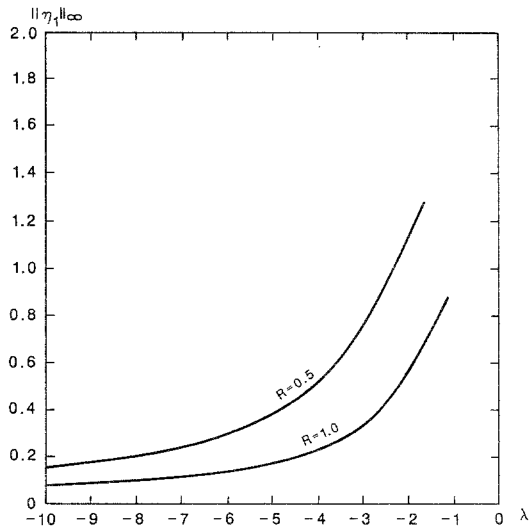


Figure 7
Subcritical solution diagrams of (41–42). The cross section of the channel is a triangle with $d = \frac{1}{4}$, $b_L = b_R = \frac{1}{2}\sqrt{16\sqrt{2} - \frac{3}{4}}$. The disturbance function $r(x)$ is

$$r(x) = \begin{cases} R\sqrt{1 - x^2} & |x| \leq 1 \\ 0, & |x| > 1. \end{cases}$$

The upper curve corresponds to $R = 1$ and at its cutting point $\lambda = \lambda_d = -1.50$. The lower curve corresponds to $R = 0.5$ and at its cutting point $\lambda = \lambda_d = -1.15$.

given by Jeffrey and Kakutani [13]. It seems that to prove this conjecture for $r \not\equiv 0$ is a very difficult problem. This is deferred to subsequent research.

Next we consider subcritical waves for which $\lambda < 0$. At the steady state, if we take constant solution of (40) upstream, then the solution downstream is cnoidal waves. Following the method due to Shen, et al. [6], we need to solve an initial value problem.

$$\lambda\eta_1 + \alpha\eta_1^2 + \beta\eta_{1xx} = r(x) + \lambda H_1 + \alpha H_1^2, \quad x > x_-, \tag{59}$$

$$\eta_1(x) = H_1, \quad x \le x_-, \tag{60}$$

$$\eta_{1x}(x) = 0, \quad x \le x_- \tag{61}$$

with the constraint

$$\frac{1}{T}\int_{x_+}^{x_+ + T} \eta_1(x)\, dx + H_1 = 0 \tag{62}$$

where $T$ is the period of cnoidal waves at downstream. The constraint (62) is used to determine $H_1 > 0$.

For a triangular channel of $d = \frac{1}{4}$, $b_R = b_L = \frac{b}{2}$, $b = \sqrt{16\sqrt{2} - \frac{3}{4}}$, and

$$r(x) = \begin{cases} R\sqrt{1 - x^2}, & |x| \le 1, \\ 0, & |x| > 1, \end{cases}$$

the numerical results are shown in Figs. 7–8. From Fig. 7, we see that there

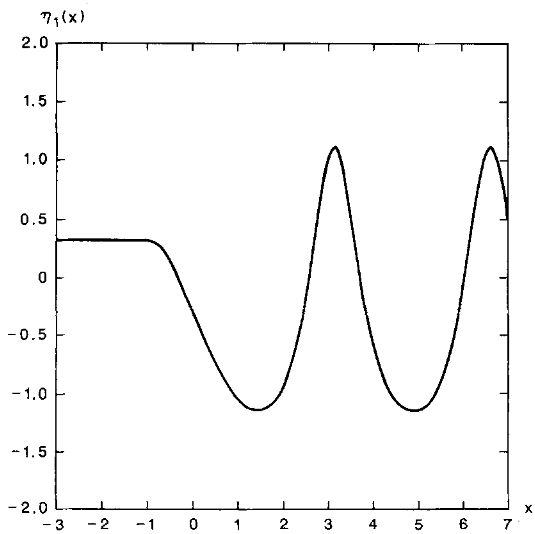


Figure 8
A subcritical solution of (52–55) as $\lambda = -2.0$ and $R = 1.0$. The period of cnoidal waves at downstream is 3.46.

exists a critical value $\lambda_d$ of $\lambda$ such that (59–62) has one solution $\lambda \le \lambda_d < 0$, and no solution for $\lambda > \lambda_d < 0$.

## Acknowledgements

The author wishes to thank Professor M. C. Shen of University of Wisconsin-Madison for his encouragement, advice, and patience during the preparation of the dissertation from which the results of this paper have been in part taken. The research has been partly supported by the Climate System Research Program, Texas A&M University under Professor G. R. North.

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### Abstract

Long waves in a current of an inviscid fluid of constant density flowing through a channel of arbitrary cross section under disturbances of pressure distribution on free surface and obstructors on the wall of the channel are considered. The first order asymptotic approximation of the elevation of the free surface satisfies a forced Korteweg-de Vries equation when the current is near its critical state. To determine the coefficients of the forced Korteweg-de Vries equation, we need to solve a linear Neumann problem of an elliptic partial differential equation, of which analytical solutions are found for constant current and rectangular or triangular cross section of the channel. It is proved that the forced Korteweg-de Vries equation has at least two solutions when the current is supercritical and the parameter $\lambda$ is greater than a critical value $\lambda_c > 0$. It is also proved that there do not exist solitary waves in a current exactly at its critical state. Numerical solutions of steady state are obtained for both supercritical and subcritical currents.

(Received: March 18, 1988, revised: August 24, 1988)