Stability of the lower cusped solitary waves

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In this Brief Communication we use numerical means to clarify a seemingly counterintuitive conclusion about the stability of multiple cusped solitary waves of locally forced Korteweg-de Vries equation (fKdV). In the context of a single layer inviscid, incompressible fluid flow over a topography, Miles, in 1986, first discovered that a cusped solitary wave can be a solution of a stationary fKdV:

\[ \eta_x - 3 \eta \eta_{xx} - 3 \eta_{xxxx} = 3^{1/6} \delta(x), \quad \eta(\pm \infty) = 0, \]  

(1)

where \( \delta \) depends on the cross sectional area of the topography [see Miles’ Eqs. (5.2) and (5.3)]. A year later, Vanden-Broeck published his discovery of two smooth solitary wave solutions for the same physical model but from the perspective of direct numerical integration of Laplace equation with a free boundary and with nonlinear boundary conditions. One of his solitary waves is higher than the other and Miles’ solitary wave is considered corresponding to the lower one. It is common knowledge that a single layer inviscid, incompressible fluid flow over a flat bottom can support a stable solitary wave as a perturbation of the free solitary wave and the lower solitary wave is considered corresponding to the lower one.

The two cusped solitary waves in question are the solutions of the following BVP:

\[ \lambda v_x + 2 \alpha v_{xx} + \beta v_{xxx} - \frac{P}{2} \delta(x), \quad v(\pm \infty) = v_x(\pm \infty) = v_{xx}(\pm \infty) = 0. \]  

(3)

The existence of these steady-state solitary wave solutions requires that the upstream Froude number \( F \) is greater than a certain value \( F_C \). The complete bifurcation diagram was given in Ref. 5. There is another special value \( F_L < 1 \). When \( F_L < F < F_C \), the solution of an initial value problem of the time dependent fKdV never approaches a stationary state. The most known phenomenon in this Froude number range is the periodic emission of solitons to upstream, discovered by Wu’s group at Caltech in 1982.

There was a great concern that in the range \( F > F_C \), among the two (or more) solitary wave solutions of the stationary fKdV, which one is stable. Because of the stability of the free solitary wave, people’s intuition might tend to suggest that the higher solitary wave is stable. In 1988, Malomed pointed out that this, as a matter of fact, is incorrect. He proved that the higher cusped solitary wave is unstable. This is an important contribution to the fKdV studies for the case of \( F > F_C \). He further conjectured that the lower solitary wave "is, to all appearance, stable" (p. 401 of Ref. 4). In the caption of his Fig. 4, his statement is that the lower solitary wave is "presumably stable."

To the authors’ knowledge, Malomed’s conjecture has not been rigorously proved. The purpose of our present work is to provide a numerical verification (not a mathematical proof) of his conjecture. Hence, our work helps with clarifying some past possible confusions about the stability of the cusped solitary waves. The serious difficulty of maintaining the cusp profile due to the strong dispersion was overcome by choosing proper time and space integrations in our semi-implicit spectral scheme.

The two cusped solitary waves in question are the solutions of the following BVP:

\[ \lambda u_x + 2 \alpha u_{xx} + \beta u_{xxx} = \frac{P}{2} \delta(x), \quad u(\pm \infty) = u_x(\pm \infty) = u_{xx}(\pm \infty) = 0. \]  

(4)

The graphics of the two solutions are shown in Fig. 1. If a stationary solution is stable, a small perturbation will not change its profile dramatically. Otherwise, it will. The stability of the stationary fKdV solitary waves (4) is defined with respect to the original time-dependent fKdV equation. Hence, we solve the IVP for the time-dependent fKdV equation.

\[ u_t + \lambda u_x + 2 \alpha u_{xx} + \beta u_{xxx} = \frac{P}{2} \delta(x). \]  

(5)
The small perturbation is introduced to the system by numerical error. For a good scheme this error can be considered as a white noise and consists of waves of all wave numbers. Since a white noise can excite any unstable mode, if the solution is stable with this type of perturbation, then it should be stable with all other types of perturbations.

Now let us briefly describe our numerical scheme. In our spectral scheme, like other spectral schemes, Eq. (5) is integrated in time by the leapfrog finite difference scheme in the spectrum space. The infinite interval is replaced by \(-L < x < L\), with \(L\) sufficiently large such that the periodicity assumption \(u(x + L, t) = u(x - L, t) = 0\) holds. The interval \((-L, L)\) is equipartitioned into \(N\) subintervals, where \(N\) is an integer power of 2.

Let us first consider the lower solitary wave for \(x_0 = -0.592 408\). We made several runs for different mesh sizes \(\Delta x\) and time steps \(\Delta t\). In all these runs we always took \(L = 30.0\). Of course, this number can be larger, but cannot be too small because of the boundary reflection of the numerical noise. We found that even with \(N = 64\) (now the space step size is \(\Delta x = 60/64 \approx 1.0\)), the cusp can still be maintained for up to \(t = 50\). In fact, this is not surprising because 64 Fourier modes can easily recover a continuous curve that is nondifferentiable at only one point. But, there is no way that a finite difference method can retain the cusp with such a large spatial step size. Certainly, when \(\Delta x\) is large, the numerical error must be large. The error is seen as large ripples in the supposedly flat region. These ripples are suppressed by increasing \(N\) and reducing \(\Delta x\). In order to see how the numerical error depends on the time step size \(\Delta t\) for a fixed \(N\), we made several runs for \(N = 128\) and \(\Delta t = 0.04, 0.02, \text{and} 0.005\). We found that the results from these four runs are almost the same. The numerical error is mainly due to the large mesh sizes \(\Delta x\). The oscillations of the ripples, due to the numerical error, appear to be slow. Hence, for a large mesh size, the numerical noise tends to be “red,” as an experienced numerical analyst would expect. When we reduce the mesh size (we have to reduce the time step as well to guarantee the stability of the numerical scheme), the scheme becomes more accurate and numerical noise becomes closer to be “white.” Among many runs we carried out for the stable lower cusped solitary wave, we feel that the following run is of satisfactory accuracy and has reasonable spatial and temporal sizes: \(N = 1024, \Delta x = \frac{\pi}{2} \approx 0.06, \text{and} \Delta t = 0.02\). We made the numerical run up to \(t = 50\). The result is shown in Fig. 2. This figure shows that the initial profile retains its original shape for a long time. Hence, the lower solitary wave is stable.

Next, let us consider the higher solitary wave for \(x_0 = -0.121 226\). Because of the rapid change of the initial profile, one would expect a large derivative in the time direction. Hence, it is necessary to choose a very small \(\Delta t\) to guarantee the accuracy (not the stability) of the scheme. The parameters are as follows: \(x_0 = -0.121 226, L = 60, N = 1024, \text{and} \Delta t = 0.002\). We made our run up to \(t = 10\). Solutions \(u(x, t = 1), u(x, t = 3), u(x, t = 5), \text{and} u(x, t = 7)\) are shown in Fig. 3. This sequence of graphics shows that the initial profile gives away some mass and gradually evolves into the smaller solitary wave. The giving away mass is included in a larger soliton moving upstream to infinity, and a small wake moving downstream to infinity. This agrees with Malomed’s qualitative conclusion on the instability of the higher solitary wave. “Development of this instability will result in establishing a one-soliton pinned state described above, while another soliton will leave for infinity” (p. 401).

To make it easy for comparison with Fig. 1, we plot \(u(x, t = 7)\) for \(x\) only in the interval \((-6, 6)\) (see Fig. 4). It clearly shows that the remaining wave still sustained on the site of forcing is the smaller solitary wave shown in Fig. 1, and that the higher solitary wave is unstable.

It is worth remarking that a numerical scheme for the nonforced KdV equation may not always work for the forced KdV equation. For example, as pointed out by Akylas, the well-known Zakusky–Kruskal and Peregrine schemes for the unforced KdV equation do not work well for the delta function forced KdV equation. Fortunately, our spectral scheme works for both forced KdV and nonforced KdV. For the spectral scheme, the ratio \(\Delta t/\Delta x\) can be relatively large, say, 0.5, and the scheme is still stable. This gain is by paying the price of doing FFT and the inverse FFT. Since it does not take many Fourier modes to reconstruct a cusp, the advantage of spectral method over a finite difference method is
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FIG. 3. The higher solitary wave is unstable. This wave dispatches some mass upstream, leaves a ripple downstream, and sustains enough mass on the forcing site. This sustained mass eventually becomes the stable (lower) solitary wave (see Fig. 4).

best reflected for solving a forced system whose solution is continuous but nondifferentiable.

The second point worthy of attention is that the KdV equation is notorious for its fast propagation of the waves of large wave numbers that are associated with the numerical noise. This problem is more serious for a nonsmooth initial profile. Conventionally, a regularized KdV equation, which filters out the fast oscillation noise, is solved, and its solution has been proved to be close to the solution of the original KdV equation. When applying our spectral scheme, this noise propagation problem does not appear to be that serious when we take sufficiently large $L$ and proper time step $\Delta t$.

For this reason, the numerical scheme here is developed for the original $fKdV$ equation rather than the regularized $fKdV$ equation.

As for the time step size, of course, $\Delta t$ cannot be too large. But, mysteriously, this $\Delta t$ should not be too small either when computing for the case of the transcritical upstream running solitons. For example, when $\lambda=0$, $\alpha=-\frac{3}{4}$, $\beta=-\frac{1}{6}$, and $P=-1$, $L=80$, and $N=512$, the optimal $\Delta t$ is around 0.125. When $\Delta t=0.05$, the result is obviously not as good as that for $\Delta t=0.125$. But, fortunately, despite the above mystery, this “proper” time step has a large range and it is quite easy to find a proper $\Delta t$.

From the above discussions, we can be confident that our numerical scheme is robust and accurate, and hence our conclusion (that lower cusped solitary wave is stable) is reliable.

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