Stability of the lower cusped solitary waves

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In this paper we provide a numerical verification of Malomed's conjecture [Physica D **32**, 393 (1988)]. Among the two cusped solitary waves of a locally forced Korteweg-de Vries equation, the lower one is stable. © *1995 American Institute of Physics.*

In this Brief Communication we use numerical means to clarify a seemingly counterintuitive conclusion about the stability of multiple cusped solitary waves of locally forced Korteweg-de Vries equation (fKdV). In the context of a single layer invicid, incompressible fluid flow over a topography, Miles, in 1986, first discovered that a cusped solitary wave can be a solution of a stationary fKdV:

$$\eta_x - 3\eta\eta_x - 3\eta_{xxx} = 3^{1/2}6\bar{\omega}\delta_x(x), \quad \eta(\pm\infty)=0, \tag{1}$$

where $\bar{\omega}$ depends on the cross section area of the topography [see Miles' Eqs. (5.2) and (5.3)[1]]. A year later, Vanden-Broeck published his discovery of two smooth solitary wave solutions for the same physical model but from the perspective of direct numerical integration of Laplace equation with a free boundary and with nonlinear boundary conditions.[2] One of his solitary waves is higher than the other and Miles' solitary wave is considered corresponding to the lower one.[3] It is common knowledge that a single layer invicid, incompressible fluid flow over a flat bottom can support a stable solitary wave, called the free solitary wave. Vanden-Broeck pointed out that the higher solitary wave can be regarded as a perturbation of the free solitary wave and the lower solitary wave as a perturbation of the trivial solution of the flat bottom model. The perturbation is, of course, due to the presence of the bump. These smooth solitary waves can be approximated by the two solitary wave solutions of a stationary forced KdV equation.[5] When the forcing has a very shot support, the forcing may be approximately expressed in terms of the Dirac delta function. Then the two solitary wave solutions of the stationary forced KdV equation can be found analytically, but each of these solitary waves has a cusp right on the location of the delta function.[4]

The existence of these steady-state solitary wave solutions requires that the upstream Froude number F is greater than a certain value $F_C>1$. The complete bifurcation diagram was given in Ref. 5. There is another special value $F_L<1$. When $F_L<F<F_C$, the solution of an initial value problem of the time dependent fKdV never approaches a stationary state. The most reknown phenomenon in this Froude number range is the periodic emission of solitons to upstream, discovered by Wu's group at Caltech in 1982.

There was a great concern that in the range $F \geqslant F_C$, among the two (or more) solitary wave solutions of the stationary fKdV, which one is stable. Because of the stability of the free solitary wave, people's intuition might tend to suggest that the higher solitary wave is stable. In 1988, Malomed pointed out that this, as a matter of fact, is incorrect. He proved that the higher cusped solitary wave is unstable.[4] This is an important contribution to the fKdV studies for the case of $F>F_C$. He further conjectured that the lower solitary wave "is, to all appearance, stable" (p. 401 of Ref. 4). In the caption of his Fig. 4, his statement is that the lower solitary wave is "presumably stable."

To the authors' knowledge, Malomed's conjecture has not been rigorously proved. The purpose of our present work is to provide a numerical verification (not a mathematical proof) of his conjecture. Hence, our work helps with clarifying some past possible confusions about the stability of the cusped solitary waves. The serious difficulty of maintaining the cusp profile due to the strong dispersion was overcome by choosing proper time and space integrations in our semi-implicit spectral scheme.

The two cusped solitary waves in questions are the solutions of the following BVP:

$$\lambda v_x + 2\alpha v v_x + \beta v_{xxx} = \frac{P}{2}\delta_x(x), \tag{2}$$

$$v(\pm\infty) = v_x(\pm\infty) = v_{xx}(\pm\infty) = 0. \tag{3}$$

When $\alpha=-\frac{3}{4}$, $\beta=-\frac{1}{6}$ (for the case of two-dimensional rectangular channel), $\lambda=3$, and $P=4$, the two cusped solitary wave solutions are given below,

$$v(x) = 6\ \mathrm{sech}^2\left(\frac{3}{2^{1/2}}(|x|-x_0)\right), \tag{4}$$

with $x_0=-0.592\,408$ (corresponding to the lower solitary wave), $x_0=-0.121\,226$ (corresponding to the higher solitary wave). The graphics of the two solutions are shown in Fig. 1.

If a stationary solution is stable, a small perturbation will not change its profile dramatically. Otherwise, it will. The stability of the stationary fKdV solitary waves (4) is defined with respect to the original time-dependent fKdV equation. Hence, we solve the IVP for the time-dependent fKdV equation,

$$u_t + \lambda u_x + 2\alpha u u_x + \beta u_{xxx} = \frac{P}{2}\delta_x(x), \tag{5}$$

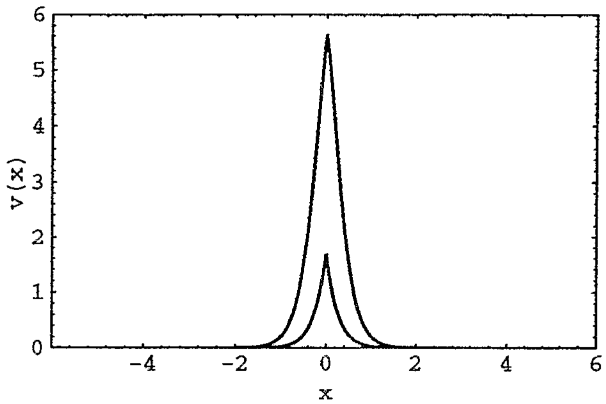

FIG. 1. Two cusped solitary wave solutions of the stationary forced KdV equation when $P=4.0$, $\lambda=3.0$, $\alpha=-\frac{3}{4}$, and $\beta=-\frac{1}{6}$.

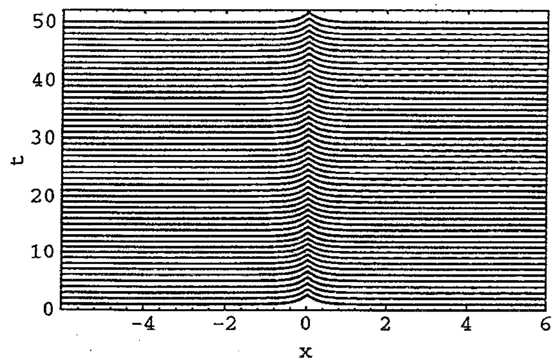

FIG. 2. The lower solitary wave is stable. Its profile does not change for a long time. The computing parameters are $L=30.0$, $N=1024$, and $\Delta t=0.02$.

$$u(x,t=0)=v(x), \quad u^{(n)}(x=\pm\infty,t=0) \quad (n=0,1,2). \tag{6}$$

The small perturbation is introduced to the system by numerical error. For a good scheme this error can be considered as a white noise and consists of waves of all wave numbers. Since a white noise can excite any unstable mode, if the solution is stable with this type of perturbation, then it should be stable with all other types of perturbations.

Now let us briefly describe our numerical scheme. In our spectral scheme, like other spectral schemes, Eq. (5) is integrated in time by the leapfrog finite difference scheme in the spectrum space. The infinite interval is replaced by $-L<x<L$, with L sufficiently large such that the periodicity assumption $u(x+L,t)=u(x-L,t)=0$ holds and FFT can be applied. The interval $(-L,L)$ is equipartitioned into N subintervals, where N is an integer power of 2.

Let us first consider the lower solitary wave for $x_0=-0.592\,408$. We made several runs for different mesh sizes Δx and time steps Δt. In all these runs we always took $L=30.0$. Of course, this number can be larger, but cannot be too small because of the boundary reflection of the numerical noise. We found that even with $N=64$ (now the space step size is $\Delta x=60/64\approx1.0$!), the cusp can still be maintained for up to $t=50$. In fact, this is not surprising because 64 Fourier modes can easily recover a continuous curve that is nondifferentiable at only one point. But, there is no way that a finite difference method can retain the cusp with such a large spatial step size. Certainly, when Δx is large, the numerical error must be large. The error is seen as large ripples in the supposedly flat region. These ripples are suppressed by increasing N and reducing Δt. In order to see how the numerical error depends on the time step size Δt for a fixed N, we made several runs for $N=128$ and $\Delta t=0.1$, 0.04, 0.02, and 0.005. We found that the results from these four runs are almost the same: The numerical error is mainly due to the large mesh sizes Δx. The oscillations of the ripples, due to the numerical error, appear to be slow. Hence, for a large mesh size, the numerical noise tends to be "red," as an experienced numerical analyst would expect. When we reduce the mesh size (we have to reduce the time step as well to guarantee the stability of the numerical scheme), the scheme becomes more accurate and numerical noise becomes closer to be "white." Among many runs we carried out for the stable lower cusped solitary wave, we feel that the following run is of satisfactory accuracy and has reasonable spatial and temporal sizes: $N=1024$, $\Delta x=\frac{60}{1024}\approx0.06$, and $\Delta t=0.02$. We made the numerical run up to $t=50$. The result is shown in Fig. 2. This figure shows that the initial profile retains its original shape for a long time. Hence, the lower solitary wave is stable.

Next, let us consider the higher solitary wave for $x_0=-0.121\,226$. Because of the rapid change of the initial profile, one would expect a large derivative in the time direction. Hence, it is necessary to choose a very small Δt to guarantee the accuracy (not the stability) of the scheme. The parameters are as follows: $x_0=-0.121\,226$, $L=60$, $N=1024$, and $\Delta t=0.002$. We made our run up to $t=10$. Solutions $u(x,t=1)$, $u(x,t=3)$, $u(x,t=5)$, and $u(x,t=7)$ are shown in Fig. 3. This sequence of graphics shows that the initial profile gives away some mass and gradually evolves into the smaller solitary wave. The giving away mass is included in a larger soliton moving upstream to infinity, and a small wake moving downstream to infinity. This agrees with Malomed's qualitative conclusion on the instability of the higher solitary wave. "Development of this instability will result in establishing a one-soliton pinned state described above, while another soliton will leave for infinity" (p. 401).

To make it easy for comparison with Fig. 1, we plot $u(x,t=7)$ for x only in the interval $(-6,6)$ (see Fig. 4). It clearly shows that the remaining wave still sustained on the site of forcing is the smaller solitary wave shown in Fig. 1, and that the higher solitary wave is unstable.

It is worth remarking that a numerical scheme for the nonforced KdV equation may not always work for the forced KdV equation. For example, as pointed out by Akylas, the reknowned Zabusky–Kruskal scheme and Peregrine scheme for the unforced KdV equation do not work well for the delta function forced KdV equation.[6] Fortunately, our spectral scheme works for both forced KdV and nonforced KdV. For the spectral scheme, the ratio $\Delta t/\Delta x$ can be relatively large, say, 0.5, and the scheme is still stable. This gain is by paying the price of doing FFT and the inverse FFT. Since it does not take many Fourier modes to reconstruct a cusp, the advantage of spectral method over a finite difference method is

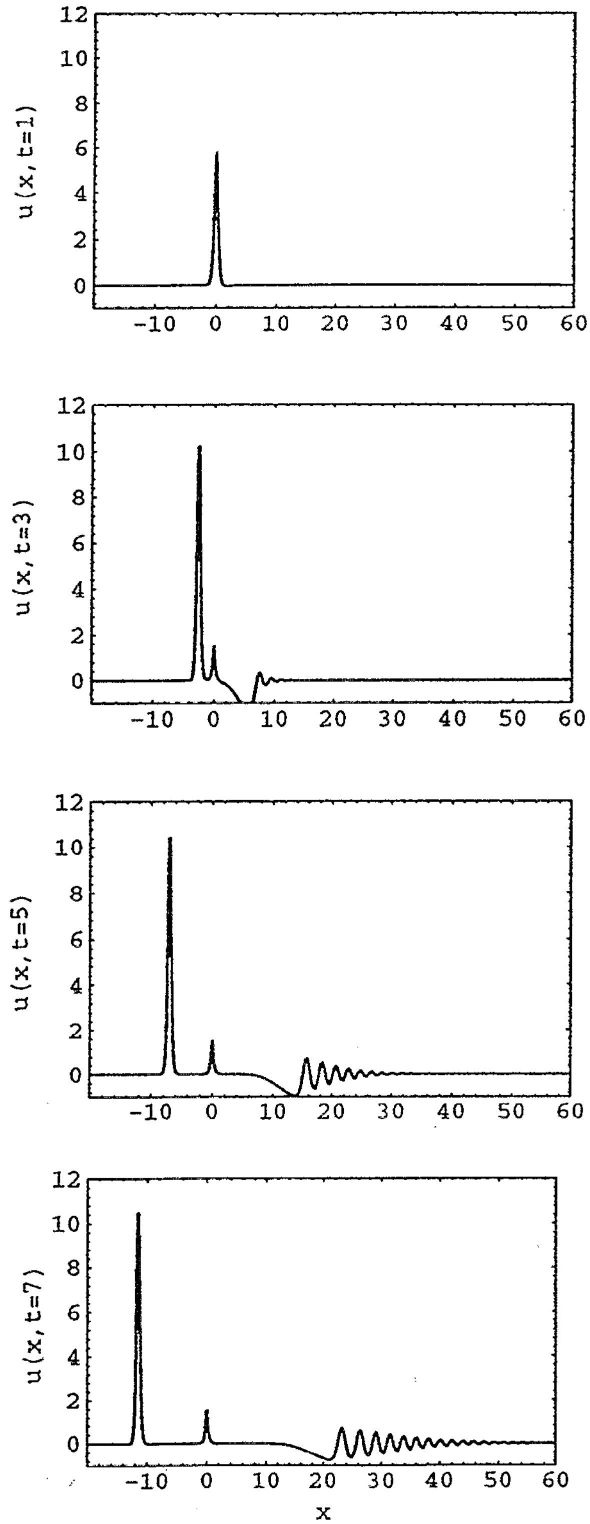

FIG. 3. The higher solitary wave is unstable. This wave dispatches some mass upstream, leaves a ripple downstream, and sustains enough mass on the forcing site. This sustained mass eventually becomes the stable (lower) solitary wave (see Fig. 4).

best reflected for solving a forced system whose solution is continuous but nondifferentiable.

The second point worthy of attention is that the KdV equation is notorious for its fast propagation of the waves of

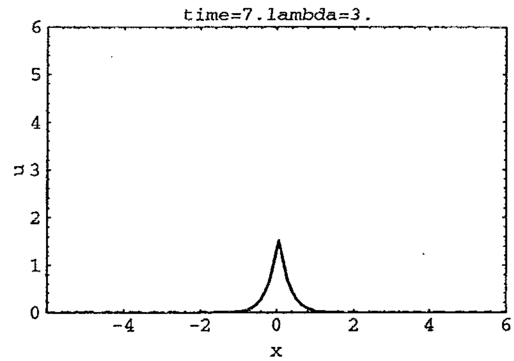

FIG. 4. The last frame in Fig. 3 is reproduced for x only in $(-6,6)$. Also see the caption of Fig. 3 for the meaning of this cusped solitary wave and compare Fig. 4 with Fig. 1.

large wave numbers that are associated with the numerical noise. This problem is more serious for a nonsmooth initial profile. Conventionally, a regularized KdV equation, which filters out the fast oscillation noise, is solved, and its solution has been proved to be close to the solution of the original KdV equation. When applying our spectral scheme, this noise propagation problem does not appear to be that serious when we take sufficiently large L and proper time step Δt. For this reason, the numerical scheme here is developed for the original fKdV equation rather than the regularized fKdV equation.

As for the time step size, of course, Δt cannot be too large. But, mysteriously, this Δt should not be too small either when computing for the case of the transcritical upstream running solitons. For example, when $\lambda=0$, $\alpha=-\frac{3}{4}$, $\beta=-\frac{1}{6}$, and $P=1$, $L=80$, and $N=512$, the optimal Δt is around 0.125. When $\Delta t=0.05$, the result is obviously not as good as that for $\Delta t=0.125$. But, fortunately, despite the above mystery, this "proper" time step has a large range and it is quite easy to find a proper Δt.

From the above discussions, we can be confident that our numerical scheme is robust and accurate, and hence our conclusion (that lower cusped solitary wave is stable) is reliable.

ACKNOWLEDGMENT

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