On a Korteweg–de Vries equation with variable coefficients in cylindrical coordinates

S. P. Shen and M. C. Shen
Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

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An equation of the Korteweg–de Vries (KdV) type with variable coefficients in cylindrical coordinates for nonlinear wave propagation on water of variable depth is presented. The special case concerning water motion with cylindrical symmetry reduces to an equation of the KdV type obtained from approximate equations for long waves.

In recent years there has been growing interest in equations of the KdV type with variable coefficients, which first appeared in the study of waves on water over a variable bottom. The equations for channels were derived by Kakutani,[1] Johnson,[2] Shuto,[3] Zhong and Shen,[4] and others, and those for ocean-like water regions were derived by Shen[5] and Prasad and Revindrau.[6] The derivation of the results in Refs. 4 and 5 is based upon a specialization of the method developed by Shen and Keller.[7] In some applications, for example, to wave propagation from an epicenter, it is desirable to express the equations in cylindrical coordinates. Recently Carbonaro *et al.*[8] have derived an equation of the KdV type for water waves with cylindrical symmetry from a system of approximate equations of the Boussinesq type given by Peregrine.[9] In this Letter we present an equation of the KdV type in terms of cylindrical coordinates obtained from the exact equations. The equation reduces to the special case considered in Ref. 8 by assuming cylindrical symmetry. We also note that the mathematical model considered here is simpler than the one in Ref. 7 and we do not include the effects of compressibility and rotation.

We consider the motion of an inviscid, incompressible fluid of constant density ρ under constant gravity g in an open water region with a variable bottom defined by $z^* = -h^*(r^*,\theta)$, where (r^*,θ^*,z^*) are the cylindrical coordinates and z^* is positive upward. Let t^* be the time, (u^*,v^*,w^*) the velocity, p the pressure, and $z^* = \eta^*(t^*,r^*,\theta)$ the free surface. Let H and L be, respectively, the vertical and horizontal scales, and we introduce the following nondimensional variables within the framework of long-wave approximation[7]:

$$r = \epsilon^{3/2} r^*/H, \quad z = z^*/H, \quad t = \epsilon^{3/2} t^*/(H/g)^{1/2},$$
$$(u,v,w) = (u^*,v^*,\epsilon^{-1/2} w^*)/(gH)^{1/2}, \quad p = p^*/(\rho g H),$$
$$h = h^*/H, \quad \eta = \eta^*/H, \quad \epsilon^{3/2} = H/L.$$

In terms of the unstarred variables, the governing equations are

$$\epsilon[(ru)_r/r + v_\theta/r] + w_z = 0, \tag{1}$$
$$\epsilon(u_t + uu_r + vu_\theta/r - v^2/r + p_r) + wu_z = 0, \tag{2}$$
$$\epsilon(v_t + uv_r + vv_\theta/r + uv/r + p_0/r) + ww_z = 0, \tag{3}$$
$$\epsilon^2(w_t + uw_r + vw_\theta/r) + \epsilon w w_z + p_z + 1 = 0; \tag{4}$$

at $z = \eta(t,r,\theta)$,

$$\epsilon(\eta_t + u\eta_r + v\eta_\theta/r) - w = 0, \tag{5}$$
$$p = 0; \tag{6}$$

at $z = -h(r,\theta)$,

$$\epsilon(uh_r + vh_\theta/r) + w = 0. \tag{7}$$

We assume that u, v, w, p, and η also depend upon a variable

$$\xi = \epsilon^{-1}\mathscr{S}(t,\theta,r),$$

where $\mathscr{S}$ is called the phase function, and each possesses an asymptotic expansion in the form

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots, \tag{8}$$

where the zeroth approximation is assumed to be given by

$$(u_0,v_0,w_0) = 0, \quad \eta_0 = 0, \quad p_0 = -z.$$

Substitution of (8) in (1)–(7) will yield a sequence of equations and boundary conditions for the successive approximations.

The equations for the first approximation are

$$\mathbf{q}_{1\xi}\cdot\mathbf{k} + w_{1z} = 0, \quad -\omega\mathbf{q}_{1\xi} + p_{1\xi}\mathbf{k} = 0, \quad p_{1z} = 0,$$
$$\omega\eta_{1\xi} + w_1 + w_1 = 0, \quad p_1 = \eta_1, \quad \text{at } z = 0;$$
$$w_1 = 0, \quad \text{at } z = -h,$$

where $\mathbf{q}_1 = (u_1,v_1)$, $\omega = -\mathscr{S}_t$, $\mathbf{k} = \nabla\mathscr{S} = \mathbf{e}_r\mathscr{S}_r + \mathbf{e}_\theta\mathscr{S}_\theta/r$, and $\mathbf{e}_r,\mathbf{e}_\theta$ are unit vectors in the r,θ directions, respectively. Assume $p_{1\xi}$ is not identically zero and $\mathbf{q}_1 \to 0$, $p_1 \to 0$ as $\xi \to \infty$. We obtain

$$\omega^2 = k^2 h = [(\mathscr{S}_r)^2 + (\mathscr{S}_\theta/r)^2]h,$$
$$\mathbf{q}_1 = \mathbf{k}\eta_1/\omega, \quad w_1 = -(k^2/\omega)\eta_{1\xi}(z+h), \tag{9}$$
$$p_1 = \eta_1(t,\xi,r,\theta).$$

Equation (9) can be solved by the associated characteristic equations, which in turn determine a two-parameter family of bicharacteristics, called rays:

$$r = r(t,\gamma_1,\gamma_2), \quad \theta = \theta(t,\gamma_1,\gamma_2),$$

where γ_1,γ_2 are constant along a ray, and

$$\frac{d\mathbf{r}}{dt} = \mathbf{e}_r\frac{dr}{dt} + \mathbf{e}_\theta r\frac{d\theta}{dt} = h\frac{\mathbf{k}}{\omega},$$

where $\mathbf{r}$ is the radial vector.

To determine η_1, we proceed to the equation for the second approximation:

$$\mathbf{q}_{2\xi}\cdot\mathbf{k} + w_{2z} = -\nabla\cdot\mathbf{q}_1,$$
$$-\mathbf{q}_{2\xi}\omega + p_{2\xi}\mathbf{k} + \mathbf{q}_{1\xi}\mathbf{q}\cdot\mathbf{k} + w_1\mathbf{q}_{1z} - \mathbf{q}_{1t} - \nabla p_1 = 0,$$
$$p_{2z} + \omega w_{1\xi} = 0,$$
$$-\eta_{2\xi}\omega + \eta_{1\xi}\mathbf{q}\cdot\mathbf{k} = w_2 - w_{1z}\eta_1 + \eta_{1t} = 0,$$
$$p_2 = \eta_2, \quad \text{at } z = 0; \quad w_2 = -\mathbf{q}_1\cdot\nabla h, \quad \text{at } z = -h.$$

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By some straightforward but a little tedious calculation, we finally obtain

$$\left(\frac{\eta_1^2}{\omega}\right)_t + \nabla\left(\eta_1^2\omega^{-1}\frac{dr}{dt}\right) + \frac{3\eta_1^2\eta_{1\xi}}{h} + \left(\frac{\omega^2 h}{3}\right)\eta_1\eta_{1\xi\xi\xi} = 0; \tag{10}$$

the detailed derivation is omitted here.[7] We make use of the relations along a ray:

$$\frac{dr\,J}{dt} = rJ\nabla\cdot\left(\frac{d\mathbf{r}}{dt}\right), \quad \frac{d\eta_1}{dt} = \frac{\partial\eta_1}{\partial t} + \left(\frac{d\mathbf{r}}{dt}\right)\cdot\nabla\eta_1,$$

where J is the Jacobian of the transformation from the ray coordinates (t,γ_1,γ_2) to (t,r,θ), and (10) becomes, along a ray,

$$\eta_{1t} + \left(\frac{1}{2}\right)\eta_1\frac{d\omega^{-1}rJ}{dt}(\omega^{-1}rJ)^{-1} + \frac{3\omega\eta_1\eta_{1\xi}}{2h} + (1/6)\,\omega^3 h\eta_{1\xi\xi\xi} = 0, \tag{11}$$

where the subscript t denotes the time derivative along a ray. At a caustic, a shoreline, and $r=0$, (11) may break down, and we defer the examination of these anomalies to a subsequent study.

Assume that h is a function of r only and all the dependent variables including $\mathscr{S}$ do not depend upon θ. We choose

$$\mathscr{S} = -t + \int_{r_0}^{r} h^{-1/2}\,dr,$$

where r_0 is some fixed radius. Then

$$\omega = -\mathscr{S}_t = 1, \quad k = \mathscr{S}_r = h^{-1/2},$$

and (11) becomes

$$\eta_{1t} + (2r)^{-1}(rh^{1/2})_r\eta_1 + \tfrac{3}{2}h^{-1}\eta_1\eta_{1\xi} + \tfrac{1}{6}h\eta_{1\xi\xi\xi} = 0.$$

We may also express the above equation in terms of r and ξ:

$$h^{1/2}\eta_{1r} + (2r)^{-1}(rh^{1/2})_r\eta_1 + \tfrac{3}{2}h^{-1}\eta_1\eta_{1\xi} + \tfrac{1}{6}h\eta_{1\xi\xi\xi} = 0,$$

where the subcript r denotes the r derivative along a ray. If we introduce a function $A(t,r,\xi)$ such that

$$\eta_1 = Ah,$$

then along a ray, we have

$$A_r + \left(\frac{3}{2}\right)\frac{AA_\xi}{h^{1/2}} + \left(\frac{h^{1/2}}{6}\right)A_{\xi\xi\xi} + \left(\frac{5}{4}h_r A h^{-1}\right) + \frac{A}{2r} = 0,$$

which is the same as derived in Ref. 8 from the approximate equations for long waves.[9]

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Influence of resistivity on energetic trapped particle-induced internal kink modes

Hamid Biglari and Liu Chen
Department of Astrophysical Sciences and Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08544

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The influence of resistivity on energetic trapped particle-induced internal kink modes, dubbed "fishbones" in the literature, is explored. A general dispersion relation, which recovers the ideal theory in its appropriate limit, is derived and analyzed. An important implication of the theory for present generation fusion devices such as the Joint European Torus [*Plasma Physics and Controlled Nuclear Fusion Research* (IAEA, London, 1984), Vol I, p.11] is that they will be stable to fishbone activity.

In the last few years there has been a great deal of theoretical activity focused on the gross magnetohydrodynamic (MHD) stability of plasma systems containing a high-energy component. The great majority of these studies have been concerned with the hot particle compressional stabilization of plasmas that would otherwise be MHD unstable. More recently, however, it has been observed that energetic particles can contribute in an altogether unfavorable way to the stability of plasma systems. This was registered in a particularly striking way in the Poloidal Divertor Experiment (PDX) heating runs at Princeton[1] where it manifested itself as periodic, sharp x-radiation bursts, increased MHD activity on the Mirnov coil system, as well as an enhanced flux of charge exchange neutral particles. These oscillations, which

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