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A SIMPLE PROOF OF THE SLOPE STABILITY THEOREM FOR ENERGY BALANCE CLIMATE MODELS

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ABSTRACT. In this paper a simple proof of the slope stability theorem for energy balance climate models is given. Mathematically, this is an interfacial free boundary value problem of a feedback reaction diffusion equation. The subtle technique of implicit differentiation in studying the stability of a regular turning point bifurcation problem is utilized and makes the proof of the slope stability theorem simpler than the existing proofs. The proof method and the derived results are useful in studying the sensitivity of the global climate to external forcings.

1. Introduction. Since the 1960s, the study of climate models has been intense. Among the simplest members of the "so-called" hierarchy of models is the energy balance model (EBM), see North, et al., for a review [**10**]. The climate state in an EBM is the earth's surface temperature, since this temperature represents the earth's most important climate state. The incoming energy from sunlight is diffused horizontally and reflected into the sky by albedo reflection. Thus the EBM is a heat conduction problem on a sphere with a latitudinally dependent heat source, the solar insolation, and linear radiative damping to space as heat sink. An important nonlinearity is introduced through the reflection of the sunlight by the earth's surface. It is assumed that when the local temperature falls below a critical value, the albedo, reflectivity of sunlight, becomes that of an ice-covered surface. This ice-albedo modification leads to a strong positive feedback mechanism.

The earth's surface is divided into two types of areas, an ice-covered area and an ice-free area. The boundary of the two areas is the so-called "ice line" and is determined by the temperature profile derived

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from the EBM and an ice-line criterion. Naturally, the polar areas are ice-covered and the equator areas are ice-free. Hence, a colder earth has stronger sunlight reflectivity, and the stronger reflectivity in turn accelerates the cooling. This positive feedback makes very large ice caps unstable. Namely, if the ice caps are so large that they reach a certain low latitude, then the ice caps will continue to grow to make the entire earth ice-covered. On the other hand, also due to albedo feedback, very small ice caps are unstable too. The earth reflectivity becomes weak when the ice caps become small. Thus, in the case of very small ice caps, the earth can absorb more solar energy and accelerate the melting of the ice caps. This phenomenon is the well-known small ice cap instability, Mengel et al. [**7**]. Therefore, when the ice line is in a moderate range and the ice caps are neither too large nor too small, a stable climate is possible.

Solar radiation is, of course, the major parameter of the ice-line position. From the above discussion, it is not hard to imagine that the latitude of the ice-line is proportional to the solar radiation when the climate is stable and inversely proportional to the solar radiation when the climate is unstable. In a diagram of the ice-line latitude versus the solar radiation, a stable climate corresponds to the section of positive slope and an unstable climate to that of negative slope. This correspondence is the slope stability theorem. The purpose of this paper is to give a simple proof of the theorem based upon the linear stability principles using implicit functions. It appears that this proof is the simplest among all the known proofs. This simple proof is a helpful tool for studying the sensitivity of the earth's climate to external forcings, such as the increase of greenhouse gases, decrease of ozone, volcano activities, and solar-radiation changes.

The content of the paper is arranged as follows. Section 2 introduces the mathematical expression of the EBM. The proof of the slope stability theorem is given in Section 3. Discussions and conclusions are in Section 4.

2. Mathematical expression of the EBM. The energy balance climate model for a zonally symmetric, nonseasonal planet, see Figure 1, is represented mathematically by the following interfacial free

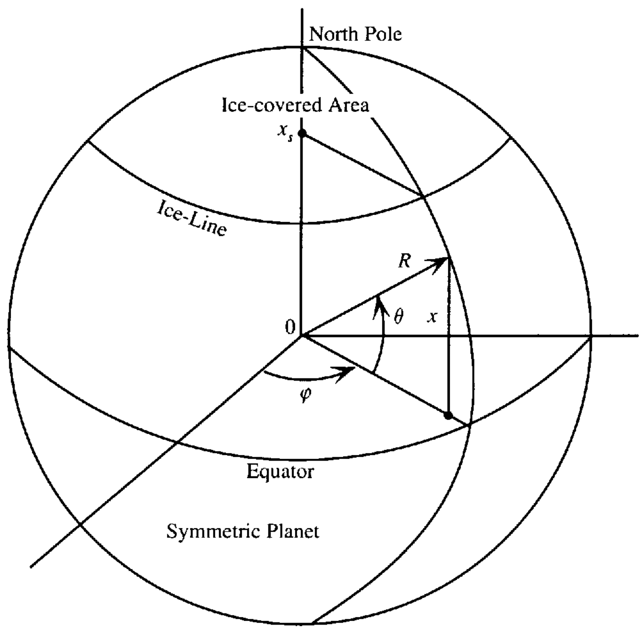

FIGURE 1. Zonally averaged, symmetric and nonseasonal planet.

boundary value problem,

$$C\frac{\partial T}{\partial t} - \frac{\partial}{\partial x}D(x)(1-x^2)\frac{\partial T}{\partial x} + A + BT = QS(x)a(x;x_s), \quad 0 < x < 1, \tag{1}$$

$$D(x)\sqrt{1-x^2}\frac{\partial T}{\partial x}\bigg|_{x=0,1} = 0. \tag{2}$$

Here C is the heat capacity per unit surface area, T is the zonally averaged surface temperature, D is the horizontal diffusion coefficient due to the eddy process and $D(x) > 0$, $A + BT$ is the energy radiation per unit surface area by Budyko's infrared law, Q is the solar constant, $S(x)$ is the mean annual normalized solar distribution function, a is the co-albedo, x is the sine of the latitude and t is time.

This idealized simple model contains no seasons. It is symmetric with respect to the equator, and no horizontal heat transport occurs through the equator. Thus, only the northern hemisphere, i.e., $x \geq 0$, is considered in the above model.

The equation

$$T(x_s, t) = T_s = \text{ constant,} \tag{3}$$

is the criterion determining the position of the ice-line $x = x_s(t)$, i.e., the interfacial free boundary. Namely, $1 \geq x > x_s$ is the ice-covered area and $0 \leq x < x_s$ is the ice-free area. The co-albedo is expressed by

$$a(x; x_s) = \begin{cases} a_i(x) & \text{if } x > x_s, \\ a_f(x) & \text{if } x < x_s. \end{cases}$$

$a(x; x_s) > 0$ and $h \equiv \lim_{x \to x_s - 0} a_f(x) - \lim_{x \to x_s + 0} a_i(x) > 0$.

A steady state solution of (1)–(2) is denoted by $T_e(x)$, then

$$T_e(x_e) = T_s \tag{4}$$

determines the equilibrium position of the ice-line $x = x_e$, i.e., the equilibrium interfacial free boundary. If we fix all the other parameters but Q in (1)–(2), then x_e is a function of Q. This relationship is represented by $G(x_s, Q) = 0$, whose graph, x_s versus Q, is called the operating curve. This curve has some regular turning points and cusp points, at which $dQ/dx_e = 0$ or dQ/dx_e does not exist. Hence x_e is a multiple valued function of Q, see Figure 2. The slope stability theorem of an equilibrium state can be stated as follows: *If the slope of the operating curve $G(x_s, Q) = 0$ has a positive (negative) slope, then the equilibrium state (T_e, x_e) is stable (unstable, respectively).*

In the last 20 years or so, the above slope stability theorem has been studied repeatedly by Held and Suarez [**5**], North [**8**], Ghil [**4**], Su and Hsieh [**11**], Drazin and Griffel [**3**], Cahalan and North [**1**], North et al. [**10**] and many others. Of course, the published papers have their own styles, use a variety of methods and make other contributions pertinent to this theorem. It appears that the proof of the slope stability theorem due to Cahalan and North [**1**] is the most complete and the most general. Nonetheless, the proof given in [**1**], like all the other published proofs of this theorem, is still unnecessarily long because the authors did not use a straightforward, standard technique in their work. This standard technique is the differentiation rule of implicit

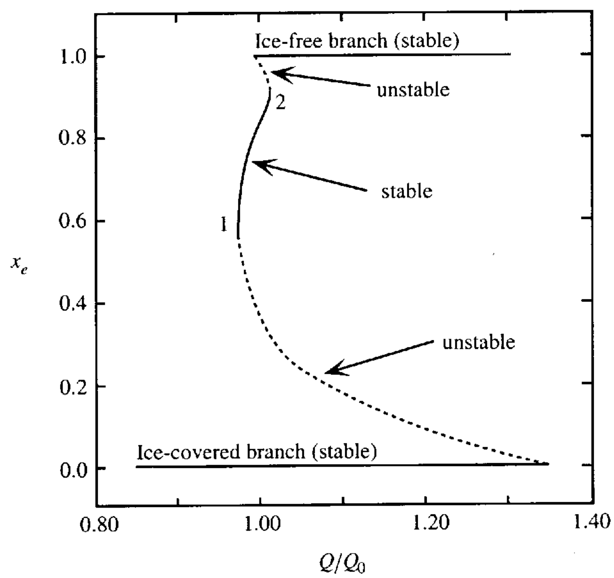

FIGURE 2. This is a reproduction of Figure 1 in [**8**]. A graph of the sine of the latitude of the equilibrium iceline, x_e, as a function of the solar constant Q (in the units of the present value Q_0). Multiple solutions correspond to a given value of the solar constant but different values of x_e.

functions [**6**]. The purpose of this brief report is to give a concise, easily understood proof of the slope stability theorem by utilizing the implicit differentiation rule.

2. Proof of the slope stability theorem. Let $(\delta T, \delta x_s)$ be a small perturbation about the equilibrium state, T_e, x_e. The equilibrium state is determined by

$$-\frac{d}{dx}k(x)\frac{dT_e}{dx} + A + BT_e = QS(x)a(x; x_e), \quad 0 < x < 1, \tag{5}$$

$$D(x)\sqrt{1-x^2}\frac{dT_e}{dx}\bigg|_{x=0,1} = 0, \tag{6}$$

$$T_e(x_e) = T_s,$$

where $k(x) = D(x)(1 - x^2)$.

In (1)–(3), let

$$T(x,t) = T_e(x) + \delta T(x,t), \tag{7}$$

$$x_s(t) = x_e + \delta x_s(t). \tag{8}$$

Substituting (7)–(8) into (1) and using (5), we have

$$C\frac{\partial}{\partial t}(\delta T) - \frac{\partial}{\partial x}k(x)\frac{\partial}{\partial x}(\delta T) + B(\delta T)$$
$$= QS(x)[a(x;x_e+\delta x_s) - a(x;x_e)].$$

In this equation the standard assumption for a linear stability problem consists of letting

$$\delta T = \theta(x)e^{-\lambda t},$$
$$\delta x_s = \eta e^{-\lambda t}.$$

Then the above partial differential equation becomes

$$-\lambda C\theta - \frac{d}{dx}k(x)\frac{d}{dx}\theta + B\theta = QS(x)[a(x;x_e+\delta x_s) - a(x;x_e)]\cdot e^{\lambda t}. \tag{9}$$

Let

$$\theta(x) = \sum_{n=1}^{\infty} H_n\varphi_n(x). \tag{10}$$

This is a standard eigenfunction expansion where the normalized eigenpairs $\{(\lambda_n,\varphi_n)\}_{n=1}^{\infty}$ are determined by

$$-\frac{d}{dx}k(x)\frac{d\varphi_n}{dx} = \lambda_n\varphi_n, \tag{11}$$

$$D(x)\sqrt{1-x^2}\frac{d\varphi_n}{dx}\bigg|_{x=0,1} = 0. \tag{12}$$

We approximate $a(x;x_e+\delta x_s)$ by $a(x;x_e) + h\delta(x-x_e)\delta x_s$, where $h = a(x_e-0;x_e) - a(x_e+0,x_e)$ and $\delta(x-x_e)$ is the Dirac delta function.

Substituting (10) into (9), multiplying the equation by $\varphi_n(x)$, and integrating the resulting equation from 0 to 1, we have

$$H_n = \frac{QS(x_e)\eta h \varphi_n(x_e)}{\lambda_n + B - \lambda C}. \tag{14}$$

Here we used

$$\int_0^1 f(x)\delta(x - x_e)\, dx = f(x_e), \quad 0 \le x_e \le 1,$$

and

$$\int_0^1 \varphi_m(x)\varphi_n(x)\, dx = \delta_{mn}.$$

Hence

$$\theta(x) = QS(x_e)\eta h \sum_{n=0}^{\infty} \frac{\varphi_n(x_e)\varphi_n(x)}{\lambda_n + B - \lambda C}. \tag{15}$$

Next we will find another simple relationship between $\theta(x_e)$ and η. From (3) and (8),

$$\begin{aligned} T_s &= T(x_e + \delta x_s, t) \\ &= T(x_e, t) + \left(\frac{\partial T}{\partial x}\right)_{x_e} \delta x_s \\ &= T_e(x_e) + \delta T(x_e, t) + T_e'(x_e)\delta x_s \\ &= T_s + \delta T(x_e, t) + T_e'(x_e)\delta x_s. \end{aligned}$$

Hence,

$$T_e'(x_e) = -\delta T(x_e, t)/\delta x_e = -\theta(x_e)/\eta. \tag{16}$$

In (5)–(6), let

$$T_e(x) = \sum_{n=0}^{\infty} T_n \varphi_n(x). \tag{17}$$

Substituting this into (5), multiplying (5) by $\varphi_n(x)$, and integrating the resulting equation from 0 to 1, we have

$$T_n = \frac{1}{\lambda_n + B}\left[- A\delta_{0n} + Q \int_0^1 S(x)a(x;x_e)\varphi_n(x)\,dx \right].$$

By (17), the ice-line criterion (4) is then

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n + B}\left[- A\delta_{0n} + Q \int_0^1 S(x)a(x;x_e)\varphi_n(x)\,dx \right]\varphi_n(x_e) = T_s. \tag{18}$$

For simplicity, we denote this by

$$G(x_e, Q) = 0. \tag{19}$$

This equation defines an implicit function of x_e versus Q. We will use the implicit differentiation rule to find dx_e/dQ, which is the slope of the operating curve.

$$\begin{aligned} \frac{\partial G}{\partial Q} &= \sum_{n=0}^{\infty} \left[\frac{1}{\lambda_n + B} \int_0^1 S(x)a(x;x_e)\varphi_n(x)\,dx \right]\varphi_n(x_e) \\ &= \frac{1}{Q}\left(T_s + \frac{A}{B}\right), \end{aligned} \tag{20}$$

and

$$\begin{aligned} \frac{\partial G}{\partial x_e} &= \sum_{n=0}^{\infty} \frac{QS(x_e)h\varphi_n^2(x_e)}{\lambda_n + B} \\ &\quad + \sum_{n=0}^{\infty} \frac{1}{\lambda_n + B}\left[-A\delta_{0n} + Q \int_0^1 S(x)a(x;x_e)\varphi_n(x)\,dx \right]\varphi_n'(x_e) \\ &= QS(x_e)h \sum_{n=0}^{\infty} \frac{\varphi_n^2(x_e)}{\lambda_n + B} + T_e'(x_e). \end{aligned} \tag{21}$$

By (16) and (15),

$$\begin{aligned} T_e'(x_e) &= -\theta(x_e)/\eta \\ &= -QS(x_e)h \sum_{n=0}^{\infty} \frac{\varphi_n^2(x_e)}{\lambda_n + B - \lambda C}. \end{aligned}$$

Hence,

$$\frac{\partial G}{\partial x_e} = -QS(x_e)h \sum_{n=0}^{\infty} \frac{C\lambda\varphi_n^2(x_e)}{(\lambda_n + B - \lambda C)(\lambda_n + B)}.$$

Therefore,

$$\begin{aligned} \frac{dQ}{dx_e} &= -\frac{\partial G}{\partial x_e} \Big/ \frac{\partial G}{\partial Q} \\ &= \frac{Q^2 S(x_e)hB}{BT_s + A} \sum_{n=0}^{\infty} \frac{C\lambda\varphi_n^2(x_e)}{(\lambda_n + B - \lambda C)(\lambda_n + B)} \\ &\equiv \Gamma(\lambda). \end{aligned} \tag{22}$$

As $\lambda \to -\infty$,

$$\begin{aligned} \Gamma(\lambda) &= \frac{Q^2 S(x_e)hB}{BT_s + A} \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n + B - \lambda C} - \frac{1}{\lambda_n + B} \right) \\ &\to -\frac{Q^2 S(x_e)hB}{BT_s + A} \sum_{n=0}^{\infty} \frac{1}{\lambda_n + B} \\ &\equiv \gamma < 0. \end{aligned}$$

By this and (22), one can graph the function $\Gamma(\lambda)$, see Figure 3. From this figure we can see that if $\gamma < (dQ/dx_e) > 0$ (< 0), then $\lambda > 0$ (there is a $\lambda < 0$, respectively). So $dQ/dx_e > 0$ (< 0), then the equilibrium state is stable (unstable, respectively).

Nonetheless, the proof is not yet finished until we have proved that

$$\frac{dQ}{dx_e} > \gamma. \tag{23}$$

This can be easily shown as follows. Since $T_e(x)$ is a strictly decreasing function of x [**1**], it follows that $T_e'(x_e) < 0$. Hence, by (21),

$$\frac{\partial G}{\partial x_e} \leq QS(x_e)h \sum_{n=0}^{\infty} \frac{\varphi_n^2(x_e)}{\lambda_n + B}.$$

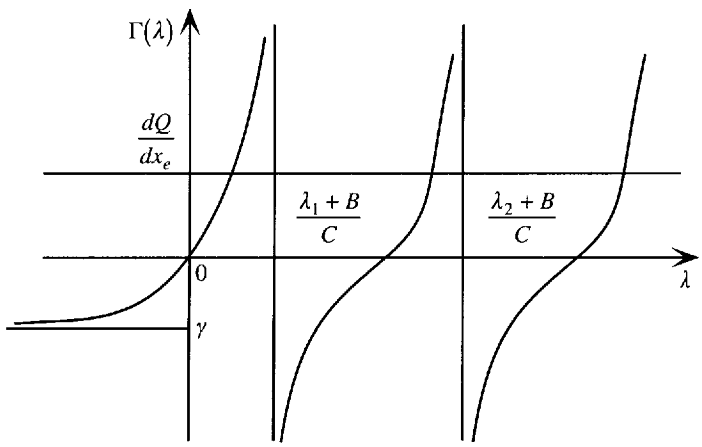

FIGURE 3. Graphic solutions of (22) for λ.

By (20), $\partial G/\partial Q > 0$ as long as $A/B + T_s > 0$. Therefore,

$$\begin{aligned}\frac{dQ}{dx_e} &= -\frac{\partial G}{\partial x_e}\Big/\frac{\partial G}{\partial Q} \\ &> -\frac{SQ(x_e)h\sum_{n=0}^{\infty}(\varphi_n^2(x_e)/(\lambda_n+B))}{(1/Q)(T_s+(A/B))} \\ &= \gamma.\end{aligned}$$

The proof is finished.

4. Discussion and conclusions. The bifurcation diagram, Figure 2, was computed by North [8] from the following parameter data,

$$\begin{aligned} D &= 0.310 Wm^{-2\circ}c^{-1} \\ A &= 201.4 Wm^{-2} \\ B &= 1.45 Wm^{-2\circ}c^{-1} \\ Q_0 &= 1337.6 Wm^{-2} \text{ (present value of the solar constant)} \\ a(x;x_s) &= \begin{cases} 0.38 & x > x_s \\ 0.68 & x < x_s. \end{cases}\end{aligned}$$

The outgoing radiation $A + BT$ may be considered as the linearization of the Stefan-Boltzmann law of black body radiation

$$\sigma(273.15 + T)^4 \approx \sigma \times 273.15^4 \times \left(1 + 4 \times \frac{T}{273.15}\right),$$

where T's unit is the Celsius degree and σ is the Stefan-Boltzmann constant. One can see that the parametrization may change with different definitions of the surface temperature T. For the same reason, the parametrization of other parameters may also change with different definitions of the surface temperature T and the complication levels of the EBM. For instance, when the land, ocean and seasons are included in the model, the radiation parameters become $A = 190.0Wm^{-2}$ and $B = 2.0Wm^{-2\circ}c^{-1}$ according to Mengel et al. [**7**].

When the global climate is at the ice-free branch, the lowest temperature, i.e., the pole temperature, is much higher than T_s. An infinitesimal perturbation of the temperature T cannot reach T_s, and hence the planet is still ice-free. Thus, the ice-free branch is stable. By a similar argument, it can be concluded that the ice-covered branch is also stable. When $0 < x_s < 1$, the slope stability theorem applies.

A remarkable feature of the diagram is the cusp near the pole [**8, 3, 9**]. From the slope stability theorem, it follows that a small ice cap on the pole is unstable. This conclusion is known by meteorologists as the SICI (Small Ice Cap Instability) [**7**]. It means that when the global climate is at the turning point 2, at which the ice cap is quite small, an infinitesimal increase of the solar constant can result in an ice free planet. Similarly, at the turning point 1, an infinitesimal decrease of the solar constant can result in an ice-covered planet.

The above dramatic conclusion has been confirmed in more comprehensive energy balance climate models. For example, the bifurcation of periodic seasonal climate models was found by Mengel et al. [**7**]. Their numerical results showed that, at a turning point of the operating curve, the small increase of the solar constant from $Q/Q_0 = 1.0822$ to $Q/Q_0 = 1.0824$ changes the summer snow-line from 70° N to a snow free planet. Something akin to the slope stability theorem seems to hold in those more complicated climate models; however, an analytic proof like the one provided here appears impossible. Only well-designed numerical simulations can be used to study the slope stability theorem. Specifically, it would be interesting to see if the slope stability theorem holds for a longitudinally dependent energy balance climate model. In this situation, the ice-line depends not only on the latitude but also the longitude. The ordinate of the operating curve, i.e., bifurcation diagram, may be the ice-covered area. It remains to be proved that the point of positive (negative) slope on the operating curve corresponds to a stable (unstable, respectively) climate.

SICI is not just a mathematical feature of energy-balance models. Crowley et al. [**2**] demonstrated that general circulation models with realistic boundary conditions can also have SICI and, in addition, the instability occurs in approximately the same parameter space as one previously found in an EBM. Their findings help explain some cases of glacial inception and abrupt transitions in the earth's history. Their studies further justify the importance of understanding the climate instability reflected in simple climate models. Reference [**2**] clearly shows the difficulty of finding SICI in complex models, and it is not known whether SICI must exist in all the reasonable climate models. When Fanning and Weaver [**12**] studied an energy-moisture balance climate model, SICI was not demonstrated. It remains unknown whether SICI exists in this energy-moisture balance model.

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