Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE DIRECT

ELSEVIER

Journal of Computational and Applied Mathematics 155 (2003) 201–222

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Finite element approximation for the viscoelastic fluid motion problem[☆]

Yinnian He[a,*], Yanping Lin[a], Samuel S.P. Shen[b], Weiwei Sun[c], Robert Tait[b]

[a] *Faculty of Science, Xi'an Jiaotong University, Xi'an 710049, People's Republic of China*
[b] *Department of Mathematics Science, University of Alberta, Edmonton, AL, Canada T6G 2G1*
[c] *Department of Mathematics, The City University of Hong Kong, Hong Kong, People's Republic of China*

Received 1 December 2001; received in revised form 12 October 2002

Abstract

In this article, the variational formulation of the two-dimensional viscoelastic fluid motion problem and its finite element approximation are considered. An local error estimate for the velocity with H^1-norm and the pressure with L^2-norm is obtained; and a uniform error estimate for the velocity and pressure with the above norms is provided if the given data satisfies the uniqueness condition.

MSC: 35L70; 65N30; 76D05; 78A10

Keywords: Viscoelastic fluid motion; Finite element; Error estimate; Regularity

1. Introduction

The Oldroyd's mathematical model of the viscoelastic fluid motion is investigated. Such a model (see [22]) can be defined by reological relation

$$k_0\sigma + k_1\frac{\partial\sigma}{\partial t} = \eta_0\xi + \eta_1\frac{\partial\xi}{\partial t}, \quad k_1\sigma(x,0) = \eta_1\xi(x,0).$$

☆ This work was supported in parts by the Special Funds for Major State Basic Research Projects G1999032801-07, NSF of China # 19971067, NSERC of Canada, RGC of Hong Kong (CityU1141/01P).

* Corresponding author. Tel.: +86-29-2669051; fax: +86-29-3237910.
E-mail address: heyn@mail.xjtu.edu.cn (Y. He).

0377-0427/03/$ - see front matter
doi:10.1016/S0377-0427(02)00864-6

Here σ is the deriator of the stress tensor and ξ is the rate deformation tensor. Namely, ξ is an $n \times n$ matrix with components

$$\xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

where $u=u(x,t)=(u_1(x,t),\dots,u_n(x,t))$ is the velocity of the fluid motion and k_0, k_1, η_0, η_1 are positive constants, and $n=2,3$. If $\eta_0 k_1 = k_0 \eta_1$ in the above relation, we shall obtain the Newton's model of incompressible viscoelastic fluid motion.

The reological relation and the motion equation in Cauchy form leads us to the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \Delta u + (u\cdot\nabla)u - \displaystyle\int_0^t \rho \exp\{-\delta(t-s)\}\Delta u \, \mathrm{d}s + \nabla p = f, \\ \operatorname{div} u = 0 (t \geqslant 0, x \in \Omega); \\ u = 0 (t \geqslant 0, x \in \partial\Omega), \ u(x,0) = u_0(x) \ (x \in \Omega); \\ (p,1) = \displaystyle\int_\Omega p(x,t)\,\mathrm{d}x = 0, \end{cases} \tag{1.1}$$

where

$$\varepsilon = \frac{\eta_1}{k_1}, \quad \rho = \frac{1}{k_1^2}(\eta_0 k_1 - k_0\eta_1), \quad \delta = \frac{k_0}{k_1},$$

Ω is an open bounded domain of points $x=(x_1,\dots,x_n)$ in R^n with smooth boundary $\partial\Omega$, $p=p(x,t)$ is the pressure of the fluid, $f=f(x,t)$ is the prescribed external force and $u_0=u_0(x)$ is the initial velocity. The last condition in (1.1) is introduced for the uniqueness of the pressure p. The problem (1.1) is the generalization of the initial-boundary value problem for the Navier–Stokes equations and is used as model in viscoelastic fluid motion (see [22,25]). We refer the readers to [17] for extensive discussions on mathematical modeling involving in memory effects for viscoelastic fluid dynamics.

The problem (1.1) has been investigated in by Oskolkov and Kotsiolis in [19], where the Ladyzhenskaja's methods were applied (see [20]). These investigations were continued in the articles of Agranovich and Sobolevskii [1–3], Sobolevskii [25,26], Orlov and Sobolev [23], and Cannon et al. [7]. The above papers dealt with the questions of existence, uniqueness and continuous dependence of the solutions upon the data. The results, local in time for $n=3$ and global in time for $n=2$, were established in [1,2,23].

The pair (u,p) is called the solution of problem (1.1) if their highest derivatives belong to $L^2([0,T];L^2(\Omega))$ for some $T>0$ (local theorem) or for arbitrary $T>0$ (nonlocal theorem), and the equations and the initial-boundary conditions are satisfied in this sense. Furthermore, an asymptotic series of the solution is constructed in [26], a spectral numerical method of the solution in the case of the periodic boundary condition is considered in [7] and a continuous backward Euler in time scheme has also been studied recently in [24].

Recently, the exponential convergence rate of $(u(x,t), p(x,t))$ to the steady-state solution $(\bar{u}(x), \bar{p}(x))$ was considered by Sobelevskii [24]. Also, the convergence to the steady state in the case of the Navier–Stokes motion (or $\rho=0$) in exterior domain was provided by Galdi et al. [8].

Here $(\bar{u}(x), \bar{p}(x))$ is a solution of boundary value problem

$$\begin{cases} -\left(\varepsilon+\dfrac{\rho}{\delta}\right)\Delta\bar{u}+(\bar{u}\cdot\nabla)\bar{u}+\nabla\bar{p}=f_\infty, \ \operatorname{div}\bar{u}=0 \ (x\in\Omega), \\ \bar{u}=0 \ (x\in\partial\Omega), \ (\bar{p},1)=0, \end{cases} \tag{1.2}$$

where $f_\infty(x)=\lim_{t\to\infty}f(x,t)$.

Remark 1.1. If the data $\nu=\varepsilon+\rho/\delta$ and f_∞ satisfies the following uniqueness condition:

$$\frac{N}{\nu^2}\|f_\infty\|_{-1}<1, \quad N=\sup_{u,v,w\in X}\frac{|b(u,v,w)|}{\|u\|\|v\|\|w\|}, \tag{1.3}$$

then problem (1.1) admits a unique solution $(\bar{u},\bar{p})\in(H^2(\Omega)^2\cap X, H^1(\Omega)\cap L_0^2(\Omega))$ such that

$$\|\bar{u}\|_2+\|\bar{p}\|_1\leqslant c\|f_\infty\|_{L^2} \quad \text{and} \quad \|\bar{u}\|\leqslant\nu^{-1}\|f_\infty\|_{-1}. \tag{1.4}$$

Here $X=H_0^1(\Omega)^2$ with the norm $\|u\|=\|\nabla u\|_{L^2}$, the trilinear form $b(u,v,w)$ is defined in Section 2,

$$\|f_\infty\|_{-1}=\sup_{v\in X}\frac{|(f_\infty,v)|}{\|v\|}$$

and $c>0$ denotes a constant depending on Ω. Hereafter, we will denote by c a generic constant which may depend on the data $(\delta,\varepsilon,\rho,\Omega)$.

Recently, He et al. [12] considered the power convergence of $(u(x,t), p(x,t))$ to $(\bar{u}(x), \bar{p}(x))$ for the two-dimensional viscoelastic fluid motion, where some important power convergence result were proved.

It is well known (see [13–15]) that it is very important to consider the error estimate uniform in time of the numerical methods for solving the nonlinear evolution partial differential equations. The error estimate uniform in time of a spectral method for solving the nonstationary Navier–Stokes equations was obtained by Heywood [13] under the assumptions about the exponential stability of a solution. The usual error estimate and error estimate uniform in time of finite element approximation of the nonstationary Navier–Stokes problem with $n=2,3$ were considered by Heywood and Rannacher [14,15]. The discrete velocity $u_h(t)$ and pressure $p_h(t)$ are determined on conforming finite element space pair (X_h,M_h) which is assumed to possess (at least) approximate properties (4.2)–(4.4). With the above statements and the smooth assumptions of the data $(u_0,f,f_t)\in(H^2(\Omega)^2\cap X, L^\infty(R^+;L^2(\Omega)^2), L^\infty(R^+;L^2(\Omega)^2)$ with $\operatorname{div}u_0=0$, finite element solution (u_h,p_h) satisfies the following error estimates:

$$\|u(t)-u_h(t)\|_{H^1}+\tau^{1/2}(t)\|p(t)-p_h(t)\|_{L^2}\leqslant\kappa e^{\kappa t}h \tag{1.5}$$

for all $t\geqslant 0$, where $\tau(t)=\min\{t,1\}$ and κ denotes a generic constant depending only on the data $(\delta,\varepsilon,\rho,\Omega,u_0,f)$. Moreover, some similar error estimates of finite element solution (u_h,p_h) for the Navier–Stokes problem with finite time t are obtained by Bernardi and Raugel [6], and Hill and Süli [16].

In this paper, our purpose is to extend the error estimates (1.5) of the finite element method to the viscoelastic fluid motion problem under the nonsmooth assumptions of the data $(u_0,f,\tau^{1/2}(t)f_t)\in(H_0^1(\Omega)^2, L^\infty(R^+;L^2(\Omega)^2), L^\infty(R^+;L^2(\Omega)^2))$ with $\operatorname{div}u_0=0$. Finite element solution (u_h,p_h) are determined on finite element space pair (X_h,M_h) which posses the approximate properties (4.2)–(4.4). The similar estimates of finite element solution (u_h,p_h) is obtained. Furthermore, we also obtain the uniform error estimates for finite element solution (u_h, p_h) if the data (v, f_∞) satisfies uniqueness condition (1.3).

Theorem 1.1. *Assume that $\rho \geqslant 0$, the assumptions (A1)–(A2) on Ω and the data (u_0, f) stated in Section 2 and the properties (4.2)–(4.4) on the finite element space pair (X_h, M_h) stated in Section 4 are valid. Then the following error estimates hold*:

$$\tau^{1/2}(t)\|u(t) - u_h(t)\|_{H^1} + \tau(t)\|p(t) - p_h(t)\|_{L^2} \leqslant \kappa e^{\kappa t} h. \tag{1.6}$$

Moreover, if the data (v, f_∞) satisfies uniqueness condition (1.3), then finite element solution (u_h, p_h) satisfies the following uniform error estimates:

$$\tau^{1/2}(t)\|u(t) - u_h(t)\|_{H^1} + \tau(t)\|p(t) - p_h(t)\|_{L^2} \leqslant \kappa h, \tag{1.7}$$

where $\kappa > 0$ denotes a generic constant which depends only on the data $(\Omega, \varepsilon, \delta, \rho, u_0, f)$.

Remark 1.2. If we set $\rho = 0$, Theorem 1.1 then gives the convergence results in the case of the Navier–Stokes flow, where the uniform error estimate (1.7) was once derived by Heywood and Rannacher in [15] under the assumption of the exact solution being exponential stable.

This paper is organized as follows. The abstract variational setting of the problem is given in Section 2, and some simple regularities of the solutions needed in the next sections are derived in Section 3, Finite element approximations of problem (1.2) is formulated in Section 4 under general assumptions of the finite element spaces. The local and uniform H^1-error estimate for discrete velocity $u_h(t)$ and the L^2-error estimate for discrete pressure $p_h(t)$ are derived in Section 5.

2. Functional setting of the viscoelastic fluid motion equations

Let Ω be a bounded domain in R^2 assumed to have a Lipschitz continuous boundary $\partial\Omega$ and to satisfy a further condition stated in (A1) below. For the mathematical setting of problems (1.1) we introduce the following Hilbert spaces

$$X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega); \int_\Omega q \, dx = 0 \right\}.$$

The spaces $L^2(\Omega)^n, n = 1, 2, 4$ are endowed with the L^2-scalar product and L^2-norm denoted by $(\cdot, \cdot)$ and $|\cdot|$. The space $H_0^1(\Omega)$ and X are equipped with their usual scalar product and norm

$$((u, v)) = (\nabla u, \nabla v), \quad \|u\| = ((u, u))^{1/2}.$$

Next, we introduce the closed subset V of X given by

$$V = \{v \in X; \operatorname{div} v = 0\}$$

and we denote by H by the closure of V in Y, i.e.

$$H = \{v \in Y; \operatorname{div} v = 0, \ v \cdot n|_{\partial\Omega} = 0\}.$$

We refer the readers to [4,6,9,11,14,27] for more details on these spaces. We usually denote the Stokes operator by $A = -P\Delta$, where P denotes the L^2-orthogonal projection of Y onto H.

As mentioned above, we need a further assumption on Ω:

(A1) Assume that the Ω is regular so that the unique solution $(v,q)\in(X,M)$ of the steady Stokes problem

$$-v\Delta v+\nabla q=g,\quad \operatorname{div} v=0 \quad \text{in } \Omega,\quad v|_{\partial\Omega}=0$$

for prescribed $g\in Y$ exists and satisfies

$$\|v\|_2+\|q\|_1\leqslant c|g|,$$

where $\|\cdot\|_i$ denotes the usual norm of Sobolev space $H^i(\Omega)$ or $H^i(\Omega)^2$, $(i=1,2)$.

We remark that the validity of assumption (A1) is known (see [14,18]) if $\partial\Omega$ is of C^2, or if Ω is a two-dimensional convex polygon. We also note that (A1) implies

$$\|v\|_2\leqslant c|Av| \quad \forall v\in D(A) \tag{2.1}$$

(see [14]). It is easily shown that

$$|v|^2\leqslant \gamma_0\|v\|^2 \quad \forall v\in X,\quad \|v\|^2\leqslant\gamma_0|Av|^2 \quad \forall v\in D(A), \tag{2.2}$$

where $D(A)=H^2(\Omega)\cap V$, γ_0 is a positive constant depending only on Ω.

Moreover, we usually make the following assumptions about the prescribed data for problem (1.1):

(A2). The data $(u_0,f,\tau^{1/2}(t)f_t)\in(V,L^\infty(R^+;Y),L^\infty(R^+;Y))$ satisfies

$$\|u_0\|+\sup_{t\geqslant 0}(|f(t)|+\tau^{1/2}(t)|f_t(t)|)\leqslant C$$

for some constants C.

Furthermore, we also introduce the bilinear operator

$$B(u,v)=(u\cdot\nabla)v+\tfrac{1}{2}(\operatorname{div} u)v \quad \forall u,v\in X.$$

We define the continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X\times X$ and $X\times M$, respectively, by

$$a(u,v)=\varepsilon((u,v)) \quad \forall u,v\in X,\ \ d(v,q)=(q,\operatorname{div} v) \quad \forall v\in X, q\in M.$$

Moreover, a trilinear form on $X\times X\times X$ is defined by

$$\begin{aligned} b(u,v,w)&=\langle B(u,v),w\rangle_{X',X}=((u\cdot\nabla)v,w)+\tfrac{1}{2}((\operatorname{div} u)v,w)\\ &=\tfrac{1}{2}((u\cdot\nabla)v,w)-\tfrac{1}{2}((u\cdot\nabla)w,v) \quad \forall u,v,w\in X.\end{aligned}$$

It is easy to verify that b satisfies the following important properties (see [4,6,10,14,27]):

$$b(u,v,w)=-b(u,w,v), \tag{2.3}$$

$$|b(u,v,w)|+|b(w,u,v)|\leqslant c(|u|^{1/2}\|u\|^{1/2}\|v\|+\|u\||v\|^{1/2}\|v\|^{1/2})|w|^{1/2}\|w\|^{1/2}, \tag{2.4}$$

$$|b(u,v,w)|\leqslant N\|u\|\|v\|\|w\| \tag{2.5}$$

for all $u,v,w\in X$, and

$$|b(v,u,w)|+|b(w,u,v)|\leqslant c\|v\|^{1/2}|Av|^{1/2}\|\|u\||w| \tag{2.6}$$

for all $u\in X, v\in D(A), w\in Y$.

With the above notations, the variational formulation of problem (1.1) reads as Find $(u, p) \in (X, M)$ such that for all $(v, q) \in (X, M)$:

$$(u_t, v) + a(u, v) + b(u, u, v) + J(t; u, v) - d(v, p) + d(u, q) = (f, v), \tag{2.7}$$

$$u(0) = u_0, \tag{2.8}$$

where

$$J(t; u, v) = \rho \left(\mathrm{e}^{-\delta t} \int_0^t \mathrm{e}^{\delta \tau} Au(\tau)\, \mathrm{d}\tau, v \right) = \rho \left(\left(\mathrm{e}^{-\delta t} \int_0^t \mathrm{e}^{\delta \tau} u(\tau)\, \mathrm{d}\tau, v \right) \right).$$

Eq. (2.7) contains the integral operator which designs the viscoelastic property. In order to proceed some theoretical analysis and numerical analysis for the variational formulation (2.7) and (2.8), we need the following useful lemmas:

Lemma 2.1. *Assume that* $s > 0$ *and* $u, v \in L^1(0, s; X)$. *Then,*

$$\int_0^s J(t; u, \mathrm{e}^{2\delta_0 t} u(t))\, \mathrm{d}t$$
$$= \frac{1}{2} \rho \mathrm{e}^{-2\alpha_0 s} \left\| \int_0^s \mathrm{e}^{\delta \tau} u(\tau)\, \mathrm{d}\tau \right\|^2 + \rho \alpha_0 \int_0^s \mathrm{e}^{-2\alpha_0 t} \left\| \int_0^t \mathrm{e}^{\delta \tau} u(\tau)\, \mathrm{d}\tau \right\|^2 \mathrm{d}t \tag{2.9}$$

holds.

Moreover, if $u, v \in L^1(0, s; D(A))$ *then,*

$$\int_0^s J(t; u, \mathrm{e}^{2\delta_0 t} Au(t))\, \mathrm{d}t$$
$$= \frac{1}{2} \rho \mathrm{e}^{-2\alpha_0 s} \left| \int_0^s \mathrm{e}^{\delta \tau} Au(\tau)\, \mathrm{d}\tau \right|^2 + \rho \alpha_0 \int_0^s \mathrm{e}^{-2\alpha_0 t} \left| \int_0^t \mathrm{e}^{\delta \tau} Au(\tau)\, \mathrm{d}\tau \right|^2 \mathrm{d}t \tag{2.10}$$

holds, and where $0 < \delta_0 < \frac{1}{2}\min\{\delta, \varepsilon/\gamma_0\}, \alpha_0 = \delta - \delta_0$.

This proof follows easily from the integration by parts, which is omitted.

Lemma 2.2. *Assume that* $s > 0$ *and* $u \in L^2(0, s; X)$. *Then, the following inequality*:

$$\alpha_0^{-1} \left\| \int_0^s \mathrm{e}^{\delta \tau} u(\tau)\, \mathrm{d}\tau \right\|^2 \mathrm{d}t + \int_0^s \mathrm{e}^{-2\alpha_0 t} \left\| \int_0^t \mathrm{e}^{\delta \tau} u(\tau)\, \mathrm{d}\tau \right\|^2 \mathrm{d}t$$
$$\leqslant \alpha_0^{-2} \int_0^s \mathrm{e}^{2\delta_0 t} \|u(t)\|^2\, \mathrm{d}t \tag{2.11}$$

holds. Moreover, if $u \in L^2(0,s;D(A))$, *then the following inequality*:

$$\alpha_0^{-1}\left|\int_0^s e^{\delta\tau}Au(\tau)\,d\tau\right|^2 dt + \int_0^s e^{-2\alpha_0 t}\left|\int_0^t e^{\delta\tau}Au(\tau)\,d\tau\right|^2 dt$$
$$\leqslant \alpha_0^{-2}\int_0^s e^{2\delta_0 t}|Au|^2\,dt \tag{2.12}$$

holds.

Proof. We prove only (2.11) since the proof of (2.12) is exactly similar. From the integration by parts, we have

$$\int_0^s e^{-2\alpha_0 t}\left\|\int_0^t e^{\delta\tau}u(\tau)\,d\tau\right\|^2 dt$$
$$= -\frac{1}{2\alpha_0}e^{-2\alpha_0 s}\left\|\int_0^s e^{\delta\tau}u(\tau)\,d\tau\right\|^2 + \frac{1}{\alpha_0}\int_0^s e^{-\alpha_0 t}\left(\left(\int_0^t e^{\delta\tau}u(\tau)\,d\tau, e^{\delta_0 t}u(t)\right)\right) dt$$
$$\leqslant \frac{1}{\alpha_0}\left(\int_0^s e^{-2\alpha_0 t}\left\|\int_0^t e^{\delta\tau}u(\tau)\,d\tau\right\|^2 dt\right)^{1/2}\left(\int_0^s e^{2\delta_0 t}\|u(t)\|^2\,dt\right)^{1/2}$$
$$-\frac{1}{2\alpha_0}e^{-2\alpha_0 s}\left\|\int_0^s e^{\delta\tau}u(\tau)\,d\tau\right\|^2,$$

which gives (2.11). □

Lemma 2.3. *Assume that* $s>0$, $\tau^{k/2}(t)u \in L^\infty(0,s;X)$ and $u_t \in L^2(0,s;Y)$. *Then*,

$$2\left|\int_0^s \tau^k(t)J(t;u,e^{2\delta_0 t}u_t(t))\,dt\right|$$
$$\leqslant \frac{\varepsilon}{8}\tau^k(s)\|e^{\delta_0 s}u(s)\|^2 + 2\frac{\rho}{\alpha_0}\left(\alpha_0+\delta+k+\frac{2\rho}{\varepsilon}\right)\int_0^s e^{2\delta_0 t}\|u\|^2\,dt. \tag{2.13}$$

Moreover, if $\tau^{k/2}(t)u \in L^\infty(0,s;D(A))$, *and* $u_t \in L^2(0,s;Y)$, *then*

$$2\left|\int_0^s \tau^k(t)J(t;u,e^{2\delta_0 t}Au_t(t))\,dt\right|$$
$$\leqslant \frac{\varepsilon}{8}\tau^k(s)|e^{\delta_0 s}Au(s)|^2 + 2\frac{\rho}{\alpha_0}\left(\alpha_0+\delta+k+\frac{2\rho}{\varepsilon}\right)\int_0^s e^{2\delta_0 t}|Au(t)|^2\,dt, \tag{2.14}$$

where $\tau(t)=\min\{t,1\}$ and $\tau^0(t)=1,\ k \geqslant 0$.

Proof. The proof of (2.13) is exactly similar to the proof of (2.14). Hence, we will only prove (2.14). Using the integration by parts, we find that

$$
\begin{aligned}
&2\int_0^s \tau^k(t)J(t;u,\mathrm{e}^{2\delta_0 t}Au_t(t))\,\mathrm{d}t\\
&=2\rho\int_0^s\left(\int_0^t \mathrm{e}^{\delta\tau}Au(\tau)\,\mathrm{d}\tau,\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-(\delta-2\delta_0)t}\tau^k(t)Au_t(t)]\right)\mathrm{d}t\\
&\quad+2\rho\int_0^s(\delta-2\delta_0)\tau^k(t)-\frac{\mathrm{d}}{\mathrm{d}t}\tau^k(t)\mathrm{e}^{-(\delta-2\delta_0)t}\left(\int_0^t \mathrm{e}^{\delta\tau}Au(\tau)\,\mathrm{d}\tau,\ Au_t(t)\right)\mathrm{d}t\\
&=2\rho\tau^k(s)\mathrm{e}^{-(\delta-2\delta_0)s}\left(\int_0^s \mathrm{e}^{\delta\tau}Au(\tau)\,\mathrm{d}\tau,Au(s)\right)-2\rho\int_0^s\tau^k(t)\mathrm{e}^{2\delta_0 t}|Au(t)|^2\,\mathrm{d}t\\
&\quad+2\rho\int_0^s((\delta-2\delta_0)\tau^k(t)-\frac{\mathrm{d}}{\mathrm{d}t}\tau^k(t))\mathrm{e}^{-(\delta-2\delta_0)t}\left(\int_0^t \mathrm{e}^{\delta\tau}Au(\tau)\,\mathrm{d}\tau,\ Au(t)\right)\mathrm{d}t\\
&=I_1(s)+I_2(s)+I_3(s).
\end{aligned}
\tag{2.15}
$$

Thus, from the Cauchy inequality, Young inequality, Lemma 2.2 and the estimate

$$0\leqslant\tau^k(t)\leqslant 1,\quad 0\leqslant\frac{\mathrm{d}}{\mathrm{d}t}\tau^k(t)\leqslant k\quad\forall t\geqslant 0,$$

it follows that

$$
\begin{aligned}
|I_1(s)|&\leqslant 2\rho\tau^k(s)\mathrm{e}^{-\alpha_0 s}\left|\int_0^s \mathrm{e}^{\delta\tau}Au(\tau)\,\mathrm{d}\tau\right||\mathrm{e}^{\delta_0 s}Au(s)|\\
&\leqslant\frac{\varepsilon}{8}\tau^k(s)|\mathrm{e}^{\delta_0 s}Au(s)|^2+\frac{4\rho^2}{\alpha_0\varepsilon}\int_0^s \mathrm{e}^{2\delta_0\tau}|Au(\tau)|^2\,\mathrm{d}\tau;\\
|I_2(s)|&=2\rho\int_0^s|\mathrm{e}^{\delta_0 t}Au(t)|^2\,\mathrm{d}t;\\
|I_3(s)|&\leqslant 2\rho(\delta+k)\left(\int_0^s \mathrm{e}^{-2\alpha_0 t}\left|\int_0^t \mathrm{e}^{\delta\tau}Au(\tau)\,\mathrm{d}\tau\right|^2\mathrm{d}t\right)^{1/2}\left(\int_0^s \mathrm{e}^{2\delta_0 t}|Au|^2\,\mathrm{d}t\right)^{1/2}\\
&\leqslant\frac{2\rho}{\alpha_0}(\delta+k)\int_0^s \mathrm{e}^{2\delta_0 t}|Au|^2\,\mathrm{d}t.
\end{aligned}
$$

Therefore the above inequalities and (2.15) yield (2.14). □

Lemma 2.4 (Gronwall lemma). *Let g,h,y be three locally integrable nonnegative functions on the time interval $[t_0,\infty)$ such that for all $t\geqslant t_0$*

$$y(t)+G(t)\leqslant C+\int_{t_0}^t h(\tau)\,\mathrm{d}\tau+\int_{t_0}^t g(\tau)y(\tau)\,\mathrm{d}\tau,\tag{2.16}$$

where $G(t)$ is a nonnegative function on $[0,\infty)$, $C \geqslant 0$ *is a constant. Then,*

$$y(t)+G(t) \leqslant \left(C+\int_{t_0}^{t} h(\tau)\,\mathrm{d}\tau\right)\exp\left(\int_{t_0}^{t} g(\tau)\,\mathrm{d}\tau\right). \tag{2.17}$$

3. Regularity

In this section, we aim to derive the regularity of the solution (u, p) of problem (1.1).

Theorem 3.1. *Suppose that assumptions* (A1)–(A2) *are valid. Then the solution* (u, p) *of problem* (1.1) *satisfies the following regularities*:

$$\|u(s)\|^2 + \mathrm{e}^{-2\delta_0 s}\int_0^s \mathrm{e}^{2\delta_0 t}(|Au|^2+\|p\|_1^2+|u_t|^2)\,\mathrm{d}t \leqslant \kappa, \tag{3.1}$$

$$\tau(s)(|Au(s)|^2+|u_t(s)|^2+\|p(s)\|_1^2)+\mathrm{e}^{-2\delta_0 s}\int_0^s \mathrm{e}^{2\delta_0 t}\tau(t)\|u_t\|^2\,\mathrm{d}t \leqslant \kappa \tag{3.2}$$

and

$$\tau^2(s)\|u_t(s)\|^2+\mathrm{e}^{-2\delta_0 s}\int_0^s \mathrm{e}^{2\delta_0 t}\tau^2(t)(|Au_t|^2+|u_{tt}|^2+\|p_t\|_1^2)\,\mathrm{d}t \leqslant \kappa \tag{3.3}$$

for all $s \geqslant 0$.

Proof. A similar argument to that used in [16,12,14] implies (3.1) and (3.2).

Moreover, differentiating (2.7) with respect to t, one finds

$$\begin{aligned}&(u_{tt},v)+\rho(Au,v)+a(u_t,v)-d(v,p_t)+d(u_t,q)\\&\quad+b(u_t,u,v)+b(u,u_t,v)=\delta J(t;u,v)+(f_t,v),\forall(v,q)\in(X,M).\end{aligned} \tag{3.4}$$

Taking $v=Au_t$ and $q=0$ in (3.4) and using the relation $\operatorname{div} Au_t=0$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u_t\|^2+\rho|Au|^2)+\varepsilon|Au_t|^2+b(u_t,u,Au_t)+b(u,u_t,Au_t)\\&\quad=\delta J(t;u,Au_t(t))+(f_t,Au_t).\end{aligned} \tag{3.5}$$

From (2.2) and (2.6), we derive

$$|(f_t,Au_t)| \leqslant \frac{\varepsilon}{8}|Au_t|^2+c|f_t|^2,$$

$$|b(u_t,u,Au_t)|+|b(u,u_t,Au_t)| \leqslant c|Au|\|u_t\||Au_t| \leqslant \frac{\varepsilon}{8}|Au_t|^2+c|Au|^2\|u_t\|^2,$$

$$|J(t,u,Au_t(t))| \leqslant \rho\mathrm{e}^{-\delta t}\left|\int_0^t \mathrm{e}^{\delta\tau}Au(\tau)\,\mathrm{d}\tau\right||Au_t(t)| \leqslant \frac{\varepsilon}{8}|Au_t|^2+c\mathrm{e}^{-2\delta t}\left|\int_0^t \mathrm{e}^{\delta_0\tau}Au(\tau)\,\mathrm{d}\tau\right|^2.$$

Hence, we obtain from (3.5) and the above inequalities that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u_t\|^2+\rho|Au|^2)+\varepsilon|Au_t|^2$$
$$\leqslant c\mathrm{e}^{-2\delta t}\left|\int_0^t \mathrm{e}^{\delta\tau}Au(\tau)\,\mathrm{d}t\right|^2+c|f_t|^2+c|Au|^2\|u_t\|^2 \quad \forall t\geqslant 0. \tag{3.6}$$

Multiplying (3.6) by $\mathrm{e}^{2\delta_0 t}\tau^2(t)$ and noting

$$0\leqslant\tau(t)\leqslant 1 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{2\delta_0 t}\tau^2(t))\leqslant 2(1+\delta_0)\tau(t),\tau(t)|Au(t)|^2\leqslant\kappa,$$

one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{2\delta_0 t}\tau^2(t)(\|u_t\|^2+\rho|Au|^2)\right]+\varepsilon\mathrm{e}^{2\delta_0 t}\tau^2(t)|Au_t|^2\leqslant c\mathrm{e}^{2\delta_0 t}\tau(t)|f_t|^2$$
$$+\kappa\mathrm{e}^{-2\alpha_0 t}\left|\int_0^t \mathrm{e}^{\delta\tau}Au(\tau)\,\mathrm{d}\tau\right|^2+\kappa\mathrm{e}^{2\delta_0 t}(\tau(t)\|u_t\|^2+|Au|^2). \tag{3.7}$$

Integrating (3.7) for t from 0 to s and using (3.1) and (3.2) and Lemma 2.2, we derive, after a final multiplication by $\mathrm{e}^{-2\delta_0 s}$, that

$$\tau^2(s)\|u_t(s)\|^2+\varepsilon\mathrm{e}^{-2\delta_0 s}\int_0^s \mathrm{e}^{2\delta_0 t}\tau^2(t)|Au_t|^2\,\mathrm{d}t\leqslant\kappa. \tag{3.8}$$

Moreover, from (3.4) and (2.6), we obtain

$$\mathrm{e}^{2\delta_0 t}\tau^2(t)|u_{tt}|^2\leqslant c\mathrm{e}^{2\delta_0 t}\tau^2(t)(|Au_t|^2+|Au|^2\|u_t\|^2)$$
$$+c\mathrm{e}^{2\delta_0 t}\tau(t)(|Au|^2+|f_t|^2)+c\mathrm{e}^{-2\alpha_0 t}\left|\int_0^t \mathrm{e}^{\delta\tau}Au(\tau)\,\mathrm{d}\tau\right|^2. \tag{3.9}$$

Next, from (2.2), (2.4)–(2.6), (3.4), (3.9) and the inf-sup condition (see [10,27]), we have

$$\tau^2(t)\mathrm{e}^{2\delta_0 t}(|u_{tt}|^2+\|p_t\|_1^2)\leqslant c\mathrm{e}^{2\delta_0 t}\tau^2(t)(|Au_t|^2+|Au|^2\|u_t\|^2)$$
$$+c\mathrm{e}^{2\delta_0 t}\tau(t)(|Au|^2+|f_t|^2)+c\mathrm{e}^{-2\alpha_0 t}\left|\int_0^t \mathrm{e}^{\delta\tau}Au(\tau)\,\mathrm{d}\tau\right|^2. \tag{3.10}$$

Integrating (3.10) from 0 to s and using (3.1) and (3.2) and Lemma 2.2, we obtain, after a final multiplication by $\mathrm{e}^{-2\delta_0 t}$, that

$$\mathrm{e}^{-2\delta_0 s}\int_0^s \mathrm{e}^{2\delta_0 t}\tau^2(t)(|u_{tt}|^2+\|p_t\|_1^2)\,\mathrm{d}t\leqslant\kappa. \tag{3.11}$$

Hence, (3.3) follows from (3.8) and (3.11). □

4. Finite element approximation

Let $h>0$ be a real positive parameter and $\tau_h=\tau_h(\Omega)$ be a uniformly regular mesh of Ω made of n-simplices K with mesh size h. We construct velocity-pressure finite element spaces $(X_h,M_h)\subset(X,M)$ based upon the mesh τ_h and define the subspace V_h of X_h given by

$$V_h=\{v_h\in X_h;\ d(v_h,q_h)=0\ \forall r_h\in M_h\}. \tag{4.1}$$

Let $P_h : Y \to V_h$ denote the L^2-orthogonal projection defined by

$$(P_h v,\ v_h) = (v, v_h) \quad \forall v \in Y,\ v_h \in X_h.$$

We assume that the couple (X_h, M_h) satisfies the following approximation properties: for each $v \in D(A)$ and $q \in H^1(\Omega) \cap M$, there exist approximations $\pi_h v \in X_h$ and $\rho_h q \in M_h$ such that

$$d(v - \pi_h v, q_h) = 0 \quad \forall q_h \in M_h,\ \|v - \pi_h v\| \leqslant ch|Av|, |q - \rho_h q| \leqslant ch\|q\|_1, \tag{4.2}$$

together with the inverse inequality

$$\|v_h\| \leqslant ch^{-1}|v_h| \quad \forall v_h \in X_h \tag{4.3}$$

and the so-called inf-sup inequality: for each $r_h \in M_h$, there exists $v_h \in X_h, v_h \neq 0$ such that

$$d(v_h, q_h) \geqslant \bar{\beta}|q_h|\|v_h\|, \tag{4.4}$$

where $\bar{\beta} > 0$ is a constant independent of h.

The following properties which are classical consequences of (4.2) and (4.3) (see [4,6,10,14,21,28]) will be very useful

$$\|P_h v\| \leqslant c\|v\| \quad \forall v \in X,\ |v - P_h v| \leqslant ch\|v - P_h v\| \quad \forall v \in X, \tag{4.5}$$

$$|v - P_h v| + h\|v - P_h v\| \leqslant ch^2|Av| \quad \forall v \in D(A), \tag{4.6}$$

$$|q - \rho_h q| \leqslant ch\|q\|_1 \quad \forall v \in H^1(\Omega) \cap M. \tag{4.7}$$

The standard finite element approximation of (2.7) and (2.8) based on (X_h, M_h) reads: Find $(u_h, p_h) \in H^1(0, T; X_h) \times L^2(0, T; M_h)$ $\forall T > 0$, such that

$$\begin{aligned}&(u_{ht}, v_h) + a(u_h, v_h) + J(t; u_h, v_h) + b(u_h, u_h, v_h) - d(v_h, p_h) + d(u_h, q_h)\\ &\quad = (f, v_h) \quad \forall (v_h, q_h) \in (X_h, M_h),\end{aligned} \tag{4.8}$$

$$u_h(0) = P_h u_0. \tag{4.9}$$

By using a similar method to ones used in [6,20,10,14,27], we can prove the following existence, uniqueness and regularity of the finite element solution (u_h, p_h).

Theorem 4.1. *Under the assumptions of Theorem* 1.1, *problem* (4.8) *and* (4.9) *possesses a unique solution* (u_h, p_h) *such that*

$$|u_h(s)|^2 + \mathrm{e}^{-2\delta_0 s}\int_0^s \mathrm{e}^{2\delta_0 t}\|u_h(t)\|^2\,\mathrm{d}t \leqslant \kappa \quad \forall s \geqslant 0, \tag{4.10}$$

$$\limsup_{s\to\infty} \|u_h(s)\| \leqslant \nu^{-1}\|f_\infty\|_{-1}. \tag{4.11}$$

Proof. By using a similar method to ones used in [6,20,10,14,27], we can prove the following existence and uniqueness of solution (u_h, p_h) for problem (4.8) and (4.9).

Moreover, by taking $(v_h, q_h) = e^{2\delta_0 t}(u_h, p_h)$ in (4.8) and using (2.3), one finds

$$\frac{1}{2}\frac{d}{dt}(e^{2\delta_0 t}|u_h(t)|^2) + \varepsilon e^{2\delta_0 t}\|u_h\|^2 + J(t; u_h, e^{2\delta_0 t}u_h(t))$$
$$= e^{2\delta_0 t}(f, u_h) + \delta_0 e^{2\delta_0 t}|u_h|^2. \tag{4.12}$$

Due to

$$|(f, u_h)| \leqslant \frac{\varepsilon}{4}\|u_h\|^2 + \frac{\gamma_0}{\varepsilon}|f|^2, \quad \frac{\varepsilon}{2}\|u_h\|^2 \geqslant \frac{\varepsilon}{2\gamma_0}|u_h|^2 \geqslant \delta_0|u_h|^2,$$

we derive from (4.12) that

$$\frac{d}{dt}(e^{2\delta_0 t}|u_h(t)|^2) + \frac{\varepsilon}{2}e^{2\delta_0 t}\|u_h\|^2 + 2J(t; u_h, e^{2\delta_0 t}u_h(t)) \leqslant \frac{2\gamma_0}{\varepsilon}|f|^2. \tag{4.13}$$

Integrating (4.13) with respect to t and using Lemma 2.1, one can obtain (4.10) after a final multiplication by $e^{-2\delta_0 t}$.

Moreover, by integrating (4.12) for t from 0 to s, one finds, after a final multiplication by $e^{-2\delta_0 s}$, that

$$|u_h(s)|^2 + 2\varepsilon e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}\|u_h(t)\|^2\,dt + 2e^{-2\delta_0 s}\int_0^s J(t; u_h, e^{2\delta_0 t}u_h(t))\,dt$$
$$= 2\delta_0 e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}|u_h(t)|^2\,dt + 2e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}(f, u_h)\,dt. \tag{4.14}$$

Letting $t \to \infty$ in (4.14) and using the L'Hospital rule and noting

$$\limsup_{s\to\infty} 2e^{-2\delta_0 s}\int_0^s J(t; u_h, e^{2\delta_0 t}u_h(t))\,dt = \frac{\rho}{\delta_0\delta}\limsup_{s\to\infty}\|u_h(s)\|^2,$$

it follows easily that

$$\left(\varepsilon + \frac{\rho}{\delta}\right)\limsup_{s\to\infty}\|u_h(s)\|^2 = \limsup_{s\to\infty}(f(s), u_h(s)) \leqslant \|f_\infty\|_{-1}\limsup_{s\to\infty}\|u_h(s)\|,$$

which yields (4.11). □

Now, we give some examples of subspaces X_h and M_h such that the assumptions (4.2)–(4.4) are satisfied. Let Ω be a convex polygonal domain and let $\{\tau_h\}, h > 0$, be a uniformly regular family of triangulations of Ω made of n-simplices K with diameters bounded by h. For any integer l, we denote by $P_l(K)$ the space of polynomials of degree less than or equal to l on K.

Example 4.1 (Girault-Raviart [10]). We set

$$X_h = \{v_h \in C^0(\bar{\Omega})^2 \cap X; v_h|_K \in P_2(K)^2 \ \forall K \in \tau_h\},$$

$$M_h = \{r_h \in M; r_h|_K \in P_0(K) \ \forall K \in \tau_h\}.$$

Example 4.2 (Bercovier-Pironneau [5]). We consider the triangulation $\tau_{h/2}$ obtained by dividing each triangle of τ_h in four triangles (by joining the mid-sides). We set

$$X_h = \{v_h \in C^0(\bar{\Omega})^2 \cap X; v_h|_K \in P_1(K)^2 \ \forall K \in \tau_{h/2}\},$$

$$M_h = \{q_h \in C^0(\bar{\Omega}) \cap M; q_h|_K \in P_1(K) \ \forall K \in \tau_h\}.$$

Lemma 4.2. *Under the assumptions of Theorem* 1.1, *the functions* $w_h = (I - P_h)u$ *and* $r_h = (I - \rho_h)p$ *satisfy for* $t \geqslant 0$,

$$h^2|w_h(t)|^2 + \mathrm{e}^{-2\delta_0 t}\int_0^t \mathrm{e}^{2\delta_0\tau}(|w_h|^2 + h^2\|w_h\|^2 + h^2|r_h|^2)\,\mathrm{d}\tau \leqslant \kappa h^4 \tag{4.15}$$

$$\tau(t)(|w_h(t)|^2 + h^2\|w_h(t)\|^2 + h^2|r_h(t)|^2) + h^2\mathrm{e}^{-2\delta_0 t}\int_0^t \mathrm{e}^{2\delta_0\tau}\tau(\tau)|w_{ht}|^2\,\mathrm{d}s \leqslant \kappa h^4; \tag{4.16}$$

$$\mathrm{e}^{-\delta_0 t}\left(\int_0^t \mathrm{e}^{2\delta_0\tau}\tau^2(\tau)(|w_{ht}|^2 + h^2\|w_{ht}(s)\|^2 + h^2|r_{ht}|^2)\,\mathrm{d}\tau\right) + h^2\tau^2(t)|w_{ht}(t)|^2 \leqslant \kappa h^4. \tag{4.17}$$

Proof. Eqs. (4.15)–(4.17) follow from (4.5)–(4.7) and Theorem 3.1. □

5. The proof of Theorem 1.1

In this section, we aim to derive the error estimates (1.6) and (1.7) for the discrete velocity u_h with H^1-norm and the discrete pressure p_h with L^2-norm stated in Theorem 1.1.

We write $u - u_h = w_h + e_h$, where $e_h = P_h u - u_h$, and $p - p_h = r_h + \mu_h$, where $\mu_h = \rho_h p - p_h$. The proof of Theorem 1.1 will be completed by combining Lemmas 5.1 and 5.2 with Lemma 5.4 below and using the norm relation:

$$\|u(t) - u_h(t)\|_{H^1}^2 = |u(t) - u_h(t)|^2 + \|u(t) - u_h(t)\|^2 \quad \forall t \geqslant 0.$$

5.1. Estimates of the velocity (I)

Lemma 5.1. *Under the assumptions of Theorem* 1.1, *the following error estimates hold for all* $s \geqslant 0$,

$$|u(s) - u_h(s)|^2 + \mathrm{e}^{-2\delta_0 s}\int_0^s \mathrm{e}^{2\delta_0 t}\|u - u_h\|^2\,\mathrm{d}t \leqslant \kappa \mathrm{e}^{\kappa s}h^2. \tag{5.1}$$

Moreover, if the data (v, f) *satisfies uniqueness condition* (1.3), *then the following uniform error estimates hold for all* $s \geqslant 0$,

$$|u(s) - u_h(s)|^2 + \mathrm{e}^{-2\delta_0 s}\int_0^s \mathrm{e}^{2\delta_0 t}\|u - u_h\|^2\,\mathrm{d}t \leqslant \kappa h^2. \tag{5.2}$$

Proof. We subtract (4.8) from (2.7) and note

$$d(w_h,q)=d(u-P_hu,q)=0 \quad \forall q\in M_h,$$

to obtain

$$(P_hu_t-u_{ht},v)+a(u-u_h,v)+J(t;u-u_h,v)-d(v,p-p_h)+d(u-u_h,q)$$
$$+b(u-u_h,u,v)+b(u,u-u_h,v)-b(u-u_h,u-u_h,v)=0 \tag{5.3}$$

for all $(v,q)\in(X_h,M_h)$. Then, setting $(v,q)=(e_h,\mu_h)=(P_hu-u_h,\rho_hp-p_h)$ in (5.3) and using (2.3), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|e_h|^2+\varepsilon\|e_h\|^2+a(w_h,e_h)+J(t;e_h+w_h,e_h(t))$$
$$+b(u,w_h,e_h)+b(e_h+w_h,P_hu,e_h)=d(e_h,r_h). \tag{5.4}$$

Due to (2.2)–(2.6), (4.5) and Theorem 3.1, one finds

$$\frac{\varepsilon}{2}\|e_h\|^2\geqslant\frac{\varepsilon}{2\gamma_0}|e_h(t)|^2\geqslant\delta_0|e_h(t)|^2,$$
$$|b(u,w_h,e_h)|+|b(w_h,P_hu,e_h)|\leqslant c\|u\|\|e_h\|\|w_h\|\leqslant\frac{\varepsilon}{16}\|e_h\|^2+\kappa\|w_h\|^2,$$
$$|b(e_h,P_hu,e_h)|\leqslant c|e_h|^{1/2}\|e_h\|^{3/2}\|P_hu\|\frac{\varepsilon}{16}\|e_h\|^2+\kappa|e_h|^2,$$
$$|a(w_h,e_h)|+|d(e_h,r_h)|\leqslant\frac{\varepsilon}{16}\|e_h\|^2+\kappa(\|w_h\|^2+|r_h|^2),$$
$$|J(t;w_h,e_h(t))|\leqslant\rho\mathrm{e}^{-\delta t}\left\|\int_0^t\mathrm{e}^{\delta\tau}w_h(\tau)\,\mathrm{d}\tau\right\|\|e_h(t)\|$$
$$\leqslant\frac{\varepsilon}{16}\|e_h\|^2+c\mathrm{e}^{-2\delta t}\left\|\int_0^t\mathrm{e}^{\delta\tau}w_h(\tau)\,\mathrm{d}\tau\right\|^2.$$

Combining the above estimates with (5.4) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}|e_h|^2+2\delta_0|e_h|^2+\frac{\varepsilon}{2}\|e_h\|^2+2J(t;e_h,e_h(t))$$
$$\leqslant\kappa\mathrm{e}^{2\delta_0t}\|w_h\|^2+c\mathrm{e}^{-2\delta t}\left\|\int_0^t\mathrm{e}^{\delta\tau}w_h(\tau)\,\mathrm{d}\tau\right\|^2+\kappa\mathrm{e}^{2\delta_0t}|e_h|^2. \tag{5.5}$$

Multiplying (5.5) by $\mathrm{e}^{2\delta_0t}$ and integrating from 0 to s and using Lemmas 2.1, 2.2 and 4.2 and the triangle inequality, we derive

$$\mathrm{e}^{2\delta_0s}|u(s)-u_h(s)|^2+\frac{\varepsilon}{2}\int_0^s\mathrm{e}^{2\delta_0t}\|u-u_h\|^2\,\mathrm{d}t$$
$$\leqslant\kappa\mathrm{e}^{2\delta_0s}h^2+\kappa\int_0^s\mathrm{e}^{2\delta_0t}|u-u_h|^2\,\mathrm{d}t. \tag{5.6}$$

Applying Lemma 2.4 to (5.6) with

$$y(t)=|\mathrm{e}^{\delta_0t}e_h(t)|^2,\quad h(t)=\kappa h^2\mathrm{e}^{2\delta_0t},\quad C=0,\quad G(t)=\frac{\varepsilon}{2}\int_0^t\mathrm{e}^{2\delta_0s}\|e_h(s)\|^2\,\mathrm{d}s,$$

we obtain, after a final multiplication by $e^{-2\delta_0 t}$, that

$$|u(s)-u_h(s)|^2 + e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}\|u-u_h\|^2\,dt \leqslant \kappa h^2 e^{\kappa s}, \tag{5.7}$$

which is (5.1).

Moreover, if the data (v, f_∞) satisfies uniqueness condition (1.3), we will give new estimates for some terms $b(u,w_h,e_h)+b(e_h+w_h,u_h,e_h)$ in (5.3) as follows:

$$|b(u,w_h,u,e_h)| + |b(w_h,u_h,e_h)| \leqslant N(\|u\|+\|u_h\|)\|w_h\|\|e_h\|,$$

$$|b(e_h,u_h,e_h)| \leqslant N\|u_h\|\|e_h\|^2.$$

Combining above estimates with (5.4) and using some previous estimates and Theorem 3.1 yields

$$\begin{aligned}
&\frac{d}{dt}|e^{\delta_0 t}e_h|^2 + 2(\varepsilon - N\|u_h\|)\|e^{\delta_0 t}e_h\|^2 + 2J(t;e_h,e^{2\delta_0 t}e_h(t))\\
&\quad\leqslant 2\delta_0 e^{2\delta_0 t}|e_h(t)|^2 + ce^{-\alpha_0 t}\left\|\int_0^t e^{\delta\tau}w_h(\tau)\,d\tau\right\|\|e_h\|\\
&\qquad+\kappa e^{2\delta_0 t}(1+\|u_h\|)(\|w_h\|+|r_h|)\|e_h\|.
\end{aligned}\tag{5.8}$$

Integrating (5.8) from 0 to s and using Lemmas 2.2 and 4.2, one finds, after a final multiplication by $e^{-2\delta_0 s}$, that

$$\begin{aligned}
&|e_h(s)|^2 + e^{-2\delta_0 s}\int_0^s 2(\varepsilon - N\|u_h\|)\|e^{\delta_0 t}e_h\|^2\,dt + 2e^{-2\delta_0 s}\int_0^s J(t;e_h,e^{2\delta_0 t}e_h(t))\,dt\\
&\quad\leqslant 2\delta_0 e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}|e_h(t)|^2\,dt + \kappa h\left(e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}(1+\|u_h\|^2)\|e_h\|^2\,dt\right)^{1/2}.
\end{aligned}\tag{5.9}$$

Letting $s\to\infty$ in (5.9), using the L'Hospital rule and Theorem 4.1, we obtain

$$(v - Nv^{-1}\|f_\infty\|_{-1})\limsup_{s\to\infty}\|e_h(s)\|^2 \leqslant \kappa h \limsup_{s\to\infty}\|e_h(s)\|. \tag{5.10}$$

Due to uniqueness condition (1.3), there holds

$$\limsup_{s\to\infty}\|e_h(s)\|^2 \leqslant \kappa h^2,$$

which together with Lemma 4.2 and (2.2) yields

$$\limsup_{s\to\infty}|u(s)-u_h(s)|^2 \leqslant \limsup_{s\to\infty}(2|w_h(s)|^2 + 2\gamma_0\|e_h(s)\|^2) \leqslant \kappa h^2. \tag{5.11}$$

Combining (5.11) with (5.1) with finite time t yields

$$|u(t)-u_h(t)|^2 \leqslant \kappa h^2 \quad \forall t \geqslant 0. \tag{5.12}$$

Substituting the estimate (5.12) into (5.6) and multiplying it by $e^{-2\delta_0 s}$, we obtain (5.2). □

5.2. Estimates of the velocity (II)

Lemma 5.2. *Under the assumptions of Theorem* 1.1, *the following error estimates hold for* $t \geqslant 0$,

$$\tau(s)\|u(s)-u_h(s)\|^2 + \mathrm{e}^{-2\delta_0 s}\int_0^s \mathrm{e}^{2\delta_0 t}\tau(t)|u_t-u_{ht}|^2\,\mathrm{d}t \leqslant \kappa \mathrm{e}^{\kappa s}h^2. \tag{5.13}$$

If the data (v, f_∞) *satisfies uniqueness condition* (1.3), *then the following uniform error estimates hold for* $t \geqslant 0$:

$$\tau(s)\|u(s)-u_h(s)\|^2 + \mathrm{e}^{-2\delta_0 s}\int_0^s \mathrm{e}^{2\delta_0 t}\tau(t)|u_t-u_{ht}|^2\,\mathrm{d}t \leqslant \kappa h^2. \tag{5.14}$$

Proof. From (4.8) and (2.7), one finds

$$\begin{aligned}&(e_{ht},v)+a(u-u_h,v)+J(t;u-u_h,v)-d(v,\mu_h)+d(e_{ht},q)-d(v,r_h)\\&\quad+b(u,u,v)-b(u_h,u_h,v)=0 \quad \forall(v,q)\in(X_h,M_h).\end{aligned} \tag{5.15}$$

By taking $v=e_{ht},\ q=\mu_h$ in (5.15), we have

$$\begin{aligned}&\frac{1}{2}\left(|e_{ht}|^2+\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u-u_h\|^2+J(t;u-u_h,u_t(t)-u_{ht}(t))\right)\\&\quad-a(u-u_h,w_{ht})-\frac{\mathrm{d}}{\mathrm{d}t}d(e_h,r_h)+d(e_h,r_{ht})\\&\quad-J(t,u-u_h,w_{ht}(t))+b(u,u,e_{ht})-b(u_h,u_h,e_{ht})=0.\end{aligned} \tag{5.16}$$

Now, let us majorize the bilinear and trilinear terms in (5.16). Thanks to (2.2), (2.6), (4.4) and (4.5), we find that

$$\begin{aligned}&b(u,u,e_{ht})-b(u_h,u_h,e_{ht})\\&\quad=b(u-u_h,u,e_{ht})+b(u,u-u_h,e_{ht})-b(u-u_h,u-u_h,e_{ht}),\end{aligned}$$

$$\begin{aligned}&|b(u-u_h,u,e_{ht})|+|b(u,u-u_h,e_{ht})|\\&\quad\leqslant c|Au|\|u-u_h\||e_{ht}| \leqslant \tfrac{1}{8}|e_{ht}|^2+c|Au|^2\|u-u_h\|^2,\end{aligned}$$

$$\begin{aligned}|b(u-u_h,u-u_h,e_{ht})| &\leqslant ch^{-1/2}\|u-u_h\|^{3/2}|u-u_h|^{1/2}|e_{ht}|\\&\leqslant \tfrac{1}{8}|e_{ht}|^2+ch^{-1}\|u-u_h\|^3|u-u_h|,\end{aligned}$$

$$\|u-u_h\| \leqslant \|u-P_hu\|+ch^{-1}|P_hu-u_h| \leqslant c\|u\|+ch^{-1}|u-u_h|,$$

$$|a(u-u_h,w_{ht})|+|d(e_h,r_{ht})| \leqslant c(\|u-u_h\|+\|w_h\|)(\|w_{ht}\|^2+|r_{ht}|),$$

$$|J(t,u-u_h,w_{ht}(t))| \leqslant c\mathrm{e}^{-\delta t}\left\|\int_0^t \mathrm{e}^{\delta\tau}(u(\tau)-u_h(\tau))\,\mathrm{d}\tau\right\|\|w_{ht}\|.$$

Combining the above estimates with (5.16) and using Theorem 3.1, we have

$$|e_{ht}|^2 + \frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\|u-u_h\|^2 - 2d(e_h,r_h)) + 2J(t;u-u_h,u_t(t)-u_{ht}(t))$$
$$\leqslant c(\|u-u_h\| + \|w_h\|)|r_{ht}| + c\mathrm{e}^{-\delta t}\left\|\int_0^t \mathrm{e}^{\delta\tau}u(\tau)-u_h(\tau)\,\mathrm{d}\tau\right\| \|w_{ht}\|$$
$$+\kappa(|Au|^2 + h^{-1}|u-u_h| + h^{-2}|u-u_h|^2)\|u-u_h\|^2. \tag{5.17}$$

Multiplying (5.17) by $\mathrm{e}^{2\delta_0 t}\tau(t)$ and using Theorem 3.1 and noting

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{2\delta_0 t}\tau(t)d(e_h,r_h))$$
$$= -\,\mathrm{e}^{2\delta_0 t}\tau(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,d(e_h,r_h) - \left(2\delta_0\tau(t) + \frac{\mathrm{d}}{\mathrm{d}t}\,\tau(t)\right)\mathrm{e}^{2\delta_0 t}d(e_h,r_h),$$

$$|d(e_h,r_h)| \leqslant c\|e_h\||r_h| \leqslant c\|u-u_h\|^2 + c\|w_h\|^2 + c|r_h|^2$$

yields

$$\tau(t)\mathrm{e}^{2\delta_0 t}|u_t-u_{ht}|^2 + \frac{\mathrm{d}}{\mathrm{d}t}(\tau(t)\mathrm{e}^{2\delta_0 t}(\varepsilon\|u-u_h\|^2 - 2d(e_h,r_h)))$$
$$+2\tau(t)J(t;u-u_h,\mathrm{e}^{2\delta_0 t}(u_t(t)-u_{ht}(t)))$$
$$\leqslant c\mathrm{e}^{2\delta_0 t}(\|w_h\|^2 + |r_h|^2 + \tau(t)|w_{ht}|^2 + \tau^2(t)\|w_{ht}\|^2 + \tau^2(t)|r_{ht}|^2)$$
$$+\kappa\mathrm{e}^{2\delta_0 t}(1 + h^{-2}|u-u_h|^2)\|u-u_h\|^2$$
$$+c\mathrm{e}^{-2\alpha_0 t}\left\|\int_0^t \mathrm{e}^{\delta\tau}(u(\tau)-u_h(\tau)\,\mathrm{d}\tau\right\|^2. \tag{5.18}$$

Integrating (5.18) with respect to t and applying Lemma 2.2, Lemmas 4.2 and 5.1, one finds, after a final multiplication by $\mathrm{e}^{-2\delta_0 s}$, that

$$\mathrm{e}^{-2\delta_0 s}\int_0^s \mathrm{e}^{2\delta_0 t}\tau(t)|u_t-u_{ht}|^2\,\mathrm{d}t + \varepsilon\tau(s)\|u(s)-u_h(s)\|^2$$
$$+2\mathrm{e}^{-2\delta_0 s}\int_0^s \tau(t)J(t;u-u_h,\mathrm{e}^{2\delta_0 t}(u_t(t)-u_{ht}(t)))\,\mathrm{d}t$$
$$\leqslant \kappa\mathrm{e}^{-2\delta_0 t}\int_0^t \mathrm{e}^{2\delta_0 s}(1+h^{-2}|u-u_h|^2)\|u-u_h\|^2\,\mathrm{d}s + \kappa h^2 + \tau(s)d(e_h(s),r_h(s)). \tag{5.19}$$

Moreover, by the application of Lemmas 4.2 and 2.3 with $k=1$, we have

$$\tau(s)d(e_h(s),r_h(s)) \leqslant c\tau(s)(\|u(s)-u_h(s)\| + \|w_h(s)\|)|r_h(s)|$$
$$\leqslant \frac{\varepsilon}{8}\,\tau(s)\|u(s)-u_h(s)\|^2 + \kappa h^2,$$

$$2e^{-2\delta_0 s}\left|\int_0^s \tau(t)J(t;u-u_h,e^{2\delta_0 t}(u_t(t)-u_{ht}(t)))\,dt\right|$$
$$\leqslant \frac{\varepsilon}{8}\tau(s)\|u(s)-u_h(s)\|^2+ce^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}\|u-u_h\|^2\,dt$$

Combining the above estimate with (5.19) yields

$$e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}\tau(t)|u_t-u_{ht}|^2\,dt+\varepsilon\tau(s)\|u(s)-u_h(s)\|^2$$
$$\leqslant \kappa h^2+\kappa e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}(1+h^{-2}|u-u_h|^2)\|u-u_h\|^2\,dt. \tag{5.20}$$

Applying Lemma 5.1 in (5.20), we have completed the proof of Lemma 5.2. □

5.3. Estimates of the Velocity (III)

Lemma 5.3. *Under the assumptions of Theorem* 1.1, *the following error estimate holds for all* $s \geqslant 0$

$$\tau^2(s)|u_t(s)-u_{ht}(s)|^2+e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}\tau^2(t)\|u_t-u_{ht}\|^2\,dt \leqslant \kappa e^{\kappa s}h^2. \tag{5.21}$$

If the data (v, f_∞) *satisfies uniqueness condition* (1.3), *then the following uniform error estimates hold for* $s \geqslant 0$:

$$\tau^2(s)|u_t(s)-u_{ht}(s)|^2+e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}\tau^2(t)\|u_t-u_{ht}\|^2\,dt \leqslant \kappa h^2. \tag{5.22}$$

Proof. Differentiating (5.3) with respect to t and using the fact:

$$d(u_t-P_h u_t,q)=0 \quad \forall q\in M_h,$$

gives

$$(e_{htt},v)+a(e_{ht}+w_{ht},v)+\rho((u-u_h,v))-d(v,\mu_{ht})+d(e_{ht},q)$$
$$+b(u_t-u_{ht},u,v)+b(u,u_t-u_{ht},v)+b(u-u_h,u_t,v)+b(u_t,u-u_h,v)$$
$$-b(u-u_h,u_t-u_{ht},v)-b(u_t-u_{ht},u-u_h,v)$$
$$=\delta J(t;u-u_h,v)+d(v,r_{ht}) \quad \forall (v,q)\in(X_h,M_h). \tag{5.23}$$

By taking $(v,q)=(e_{ht},\mu_{ht})$ in (5.23) and using (2.3), we have

$$\frac{1}{2}\frac{d}{dt}|e_{ht}|^2+\varepsilon\|e_{ht}\|^2+a(w_{ht},e_{ht})+\rho((u-u_h,e_{ht}))+b(u_t-u_{ht},u,e_{ht})$$
$$+b(u,w_{ht},e_{ht})+b(u-u_h,u_t,e_{ht})+b(u_t,u-u_h,e_{ht})$$
$$-b(u-u_h,w_{ht},e_{ht})-b(e_{ht}+w_{ht},u-u_h,e_{ht})$$
$$=\delta J(t;u-u_h,e_{ht})+d(e_{ht},r_{ht}). \tag{5.24}$$

Now, let us majorize the bilinear and trilinear terms in (5.24). Thanks to (2.3)–(2.6), we find that

$$|a(w_{ht}, e_{ht})| + |d(e_{ht}, r_{ht})| \leqslant \frac{\varepsilon}{16}|e_{ht}|^2 + c(\|w_{ht}\|^2 + |r_{ht}|^2),$$

$$\rho((u-u_h, e_{ht}))| \leqslant \frac{\varepsilon}{16}\|e_{ht}\|^2 + c\|u-u_h\|^2,$$

$$b(u_t - u_{ht}, u, e_{ht})| + |b(u, u_t - u_{ht}, e_{ht})|$$
$$\leqslant c|Au||e_{ht}||u_t - u_{ht}| \leqslant \frac{\varepsilon}{8}\|e_{ht}\|^2 + c|Au|^2|u_t - u_{ht}|^2,$$

$$|b(u-u_h, u_t, e_{ht})| + |b(u_t, u-u_h, e_{ht})|$$
$$\leqslant c\|u_t\|\|u-u_h\|\|e_{ht}\| \leqslant \frac{\varepsilon}{16}\|e_{ht}\|^2 + c\|u_t\|^2\|u-u_h\|^2,$$

$$|b(w_{ht}, u-u_h, e_{ht})| + |b(u-u_h, w_{ht}, e_{ht})|$$
$$\leqslant c\|u-u_h\|\|w_{ht}\|\|e_{ht}\| \leqslant \frac{\varepsilon}{16}\|e_{ht}\|^2 + c(\|u\|^2 + h^{-2}|u-u_h|^2)\|w_{ht}\|^2,$$

$$|b(e_{ht}, u-u_h, e_{ht})| \leqslant c|e_{ht}|^{1/2}\|e_{ht}\|^{3/2}|u-u_h|^{1/2}\|u-u_h\|^{1/2} \leqslant \frac{\varepsilon}{16}\|e_{ht}\|^2$$
$$+c|u-u_h|^2(\|u\|^2 + h^{-2}|u-u_h|^2)(|u_t - u_{ht}|^2 + |w_{ht}|^2),$$

$$|J(t; u-u_h, e_{ht}(t))| \leqslant \frac{\varepsilon}{16}\|e_{ht}\|^2 + c\mathrm{e}^{-2\delta_0 t}\int_0^t \mathrm{e}^{2\delta_0\tau}\|u-u_h\|^2\,\mathrm{d}\tau.$$

Combining above estimates with (5.24) and applying Theorems 3.1 and 4.1 yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}|e_{ht}|^2 + e\|u_t - u_{ht}\|^2 \\
&\quad\leqslant c(1+\|u_t\|^2)\|u-u_h\|^2 + (1+h^{-2}|u-u_h|^2)(\|w_{ht}\|^2 + |r_{ht}|^2) \\
&\quad\quad+\kappa(1+|Au|^2 + h^{-2}|u-u_h|^2)|u_t - u_{ht}|^2 \\
&\quad\quad+c\mathrm{e}^{-2\delta t}\left\|\int_0^t \mathrm{e}^{\delta\tau}(u(\tau)-u_h(\tau))\,\mathrm{d}\tau\right\|^2.
\end{aligned} \tag{5.25}$$

Multiplying (5.25) by $\mathrm{e}^{2\delta_0 t}\tau^2(t)$ and applying Theorem 3.1 and noting

$$\tau^2(t)\|u_t(t)\|^2 \leqslant \kappa, \quad \tau(t)|Au|^2 \leqslant \kappa,$$

we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\tau^2(t)\mathrm{e}^{2\delta_0 t}|e_{ht}|^2) + \varepsilon\tau^2(t)\mathrm{e}^{2\delta_0 t}\|u_t - u_{ht}\|^2 \\
&\quad\leqslant c\mathrm{e}^{-2\alpha_0 t}\left\|\int_0^t \mathrm{e}^{\delta\tau}(u(\tau)-u_h(\tau))\,\mathrm{d}\tau\right\|^2 + \kappa\mathrm{e}^{2\delta_0 t}(1+h^{-2}|u-u_h|^2)\|u-u_h\|^2 \\
&\quad\quad+\kappa\mathrm{e}^{2\delta_0 t}(1+h^{-2}|u-u_h|^2)(\tau(t)|u_t - u_{ht}|^2 + \tau^2(t)(\|w_{ht}\|^2 + |r_{ht}|^2)).
\end{aligned} \tag{5.26}$$

Integrating (5.26) from 0 to s and applying Lemmas 2.2 and 4.2, we derive

$$\tau^2(s)\mathrm{e}^{2\delta_0 s}|u_t(s)-u_{ht}(s)|^2+\varepsilon\int_0^s\tau^2(t)\mathrm{e}^{2\delta_0 t}\|u_t-u_{ht}\|^2\,\mathrm{d}t$$
$$\leqslant \kappa\mathrm{e}^{2\delta_0 s}\left(h^2+\sup_{0\leqslant t\leqslant s}|u(t)-u_h(t)|^2\right)+c\int_0^s\mathrm{e}^{2\delta_0 t}\|u-u_h\|^2\,\mathrm{d}t$$
$$+\kappa\int_0^s\mathrm{e}^{2\delta_0 t}(1+h^{-2}|u-u_h|^2)(\|u-u_h\|^2+\tau(t)|u_t-u_{ht}|^2)\,\mathrm{d}t. \tag{5.27}$$

Applying Lemmas 5.1 and 5.2 in (5.27) and multiplying it by $\mathrm{e}^{-2\delta_0 s}$, we have completed the proof of Lemma 5.3. □

5.4. Estimates of pressure

We are now ready to give the error estimate of the pressure. The inf-sup condition (4.4) guarantees that

$$|\mu_h(t)|\leqslant c\sup_{0\neq v\in X_h}\frac{d(v,\mu_h(t))}{\|v\|}, \tag{5.28}$$

where, due to (5.3),

$$d(v,\mu_h(t))=(u_t(t)-u_{ht}(t),v)+a(u(t)-u_h(t),v)-d(v,r_h)$$
$$+J(t;u-u_h,v)+b(u-u_h,u,v)$$
$$+b(u,u-u_h,v)-b(u-u_h,u-u_h,v)\quad\forall v\in X_h. \tag{5.29}$$

In view of (2.5), Theorems 3.1 and 4.1 and (5.28) and (5.29), one finds

$$\tau(t)|\mu_h(t)|\leqslant c(\tau(t)|u_t(t)-u_{ht}(t)|+\tau^{1/2}(t)(1+h^{-1}|u-u_h|)\|u(t)-u_h(t)\|$$
$$+\tau^{1/2}(t)|r_h(t)|)+c\mathrm{e}^{-\delta_0 t}\left(\int_0^t\mathrm{e}^{2\delta_0\tau}\|u(\tau)-u_h(\tau)\|^2\,\mathrm{d}\tau\right)^{1/2}\quad\forall t\geqslant 0. \tag{5.30}$$

Applying Lemmas 5.1–5.3 and Lemma 4.2 in (5.30) and the triangle inequality, we obtain the following error estimates.

Lemma 5.4. *Under the assumptions of Theorem* 1.1, *the following error estimate holds*:

$$\tau(t)|p(t)-p_h(t)|\leqslant\kappa\mathrm{e}^{\kappa t}h\quad\forall t\geqslant 0. \tag{5.31}$$

If the data (v,f_∞) *satisfies uniqueness condition* (1.3), *then the following uniform error estimates hold*:

$$\tau(t)|p(t)-p_h(t)|\leqslant\kappa h\quad\forall t\geqslant 0. \tag{5.32}$$

Acknowledgements

The authors would like to thank the editor and referees for their valuable comments and suggestions which helped to improve the results of this paper.

References

[1] Yu.Ya. Agranovich, P.E. Sobolevskii, Investigation of viscoelastic fluid mathematical model, RAC.Ukranian SSR., Ser. A 10 (1989) 71–73.
[2] Yu.Ya. Agranovich, P.E. Sobolevskii, The motion of non-linear viscoelastic fluid, RAC. USSR 314 (1990) 231–233.
[3] Yu.Ya. Agranovich, P.E. Sobolevskii, Motion of non-linear viscoelastic fluid, Nonlinear Anal. 32 (1998) 755–760.
[4] A. Ait Ou Ammi, M. Marion, Nonlinear Galerkin methods and mixed finite elements: two-grid algorithms for the Navier–Stokes equations, Numer. Math. 68 (1994) 189–213.
[5] J. Bercovier, O. Pironneau, Error estimates for finite element solution of the Stokes problem in the primitive variables, Numer. Math. 33 (1979) 211–224.
[6] C. Bernardi, G. Raugel, A conforming finite element method for the time dependent Navier–Stokes equations, SIAM J. Numer. Anal. 22 (1985) 455–473.
[7] J.R. Cannon, R.E. Ewing, Yinnian He, Yanping Lin, A modified nonlinear Galerkin method for the viscoelastic fluid motion equations, Internat. J. Eng. Sci. 37 (1999) 1643–1662.
[8] G.P. Galdi, J.G. Heywood, Y. Shibata, On the global existence and convergence to steady state of Navier–Stokes flow past an obstacle that is started from rest, Arch. Rational Mech. Anal. 138 (1977) 307–318.
[9] B. Garcia-Archilia, E.S. Titi, Postprocessing the Galerkin method: the finite element case, SIAM J. Numer. Anal. 37 (1995) 470–499.
[10] V. Girault, P.A. Raviart, Finite Element Approximation of the Navier–Stokes Equations, Springer, Berlin, Heidelberg, New York, 1981.
[11] Y.N. He, K.T. Li, Convergence and stability of finite element nonlinear Galerkin method for the Navier–Stokes equations, Numer. Math. 79 (1998) 77–106.
[12] Y.N. He, Y. Lin, S. Shen, R. Tait, Convergence to steady state of the viscoelastic fluid motion, Adv. Differential Equations 7 (2002) 717–742.
[13] J.G. Heywood, An error estimate uniform in time for spectral Galerkin approximations of the Navier–Stokes problem, Pacific J. Math. 98 (1982) 333–345.
[14] J.G. Heywood, R. Rannacher, Finite element approximation of the nonstationary Navier–Stokes problem I: regularity of solutions and second-order error estimates for spatial discretization, SIAM J. Numer. Anal. 19 (1982) 275–311.
[15] J.G. Heywood, R. Rannacher, Finite element approximation of the nonstationary Navier–Stokes problem II: stability of solutions and error estimates uniform in time, SIAM J. Numer. Anal. 23 (1986) 750–777.
[16] A.T. Hill, E. Süli, Approximation of the global attractor for the incompressible Navier–Stokes equations, IMA J. Numer. Anal. 20 (2000) 633–667.
[17] D. Joseph, Fluid Dynamics of Viscoelastic Liquids, Springer, New York, NY, 1990.
[18] R.B. Kellogg, J.E. Osborn, An regularity result for the Stokes problem in a convex polygon, J. Funct. Anal. 21 (1976) 397–431.
[19] A.A. Kotsiolis, A.P. Oskolkov, On the solvability of fundamental initial-boundary value problem for the motion equations of Oldroy's fluid and the behavior of solutions, when $t \to \infty$, Notes Sci. LOMI 150 (1986) 48–52.
[20] O.A. Ladyzenskaja, Mathematical questions of viscous non-compressible fluid dynamics, M. Nauka 1970, 288pp.
[21] M. Marion, J. Xu, Error estimates on a new nonlinear Galerkin method based on two-grad finite elements, SIAM J. Numer. Anal. 32 (1995) 1170–1184.
[22] J.G. Oldroyd, Reologia-Mir, 1962, pp. 805.
[23] V.P. Ortov, P.E. Sobolevskii, On mathematical models of a viscoelastic with a memory, Differential Integral Equations 4 (1991) 103–115.
[24] A. Pani, J. Yuan, P. Damazio, On linearized backward Euler method for the equations of motion arising in the Oldroyd model, preprint.

[25] P.E. Sobolevskii, Stabilization of viscoelastic fluid motion, Oldroyd's mathematical model, Differential Integral Equations 7 (1994) 1579–1621.
[26] P.E. Sobolevskii, Asymptotics of stable viscoelastic fluid motion, Oldroyd's mathematical model, Math. Nachr. 177 (1996) 281–305.
[27] R. Temam, Navier–Stokes Equations, Theory and Numerical Analysis, 3rd Edition, North-Holland, Amsterdam, 1983.
[28] J. Xu, Two-grad discretization techniques for linear and nonlinear PDEs, SIAM J. Numer. Anal. 33 (1996) 1759–1777.