Advances in Differential Equations Volume 7, Number 6, June 2002, Pages 717–742

ON THE CONVERGENCE OF VISCOELASTIC FLUID FLOWS TO A STEADY STATE*

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(Submitted by: Viorel Barbu)

Abstract. The initial-boundary value problems describing motion of a two-dimensional viscoelastic fluid are investigated by using the methods of variational formulation and inequality estimates. Both the exponential and power convergence of the solutions to a steady state solution of the viscoelastic fluid flows are proved under prescribed conditions. The convergences to a stead state solution of the Navier-Stokes flows is a special case of the results.

1. Introduction

This paper investigates the convergence of unsteady viscoelastic fluid flows to a steady state flow as $t \to \infty$. The unsteady viscoelastic fluid flows are governed by the Oldroyd's mathematical model. Such a model (see [9]) can be defined by the rheological relation

$$k_0\sigma + k_1\frac{\partial\sigma}{\partial t} = \eta_0\xi + \eta_1\frac{\partial\xi}{\partial t},\ k_1\sigma(x,0) = \eta_1\xi(x,0). \tag{1.1}$$

Here σ is the deviator of the stress tensor and ξ is the strain tensor. Namely, ξ is an $m \times m$ matrix with components

$$\xi_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}),$$

*This paper is supported by The National Key Major Program for Developing Basic Sciences #G1999032801, National Natural Science Foundation of China, Natural Science Foundation of Shaanxi Province of China, Overseas Chinese Well-known Scholars Program of the Chinese Academy of Sciences, and Natural Sciences and Engineering Research Council of Canada.

Accepted for publication: November 2001.

AMS Subject Classifications: 35L70, 76D05, 78A10.

where $u = u(x,t) = (u_1(x,t), \cdots, u_m(x,t))$ is the velocity of the fluid motion and $k_0,\ k_1, \eta_0,\ \eta_1$ are positive constants, $m = 2, 3$. If $\eta_0 k_1 = k_0 \eta_1$ in (1.1), we shall obtain the Newton's model of incompressible viscoelastic fluid motion.

Relation (1.1) and the motion equation in the Cauchy form leads to the following initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \epsilon \Delta u + (u \cdot \nabla) u - \displaystyle\int_0^t \rho \exp\{-\delta(t-s)\} \triangle u ds + \nabla p = f, \\ \text{div } u = 0 \quad (t \geq 0, x \in \Omega); \\ u = 0 \quad (t \geq 0, x \in \partial\Omega); \\ u(x,0) = u_0 \quad (x \in \Omega); \\ (p,1) = \displaystyle\int_\Omega p(x,t) dx = 0, \end{cases} \tag{1.2}$$

where

$$\epsilon = \frac{\eta_1}{k_1}, \quad \rho = \frac{1}{k_1^2}(\eta_0 k_1 - k_0 \eta_1), \quad \delta = \frac{k_0}{k_1},$$

Ω is an open but bounded domain of points $x = (x_1, \cdots, x_m)$ in R^m with smooth boundary $\partial\Omega$, $p = p(x,t)$ is the pressure of the fluid, $f = f(x,t)$ is the prescribed external force and $u_0 = u_0(x)$ is the initial velocity. The last condition in (1.2) is introduced for the uniqueness of the pressure p. Problem (1.2) is the generalization of the initial-boundary value problem for the Navier-Stokes equations.

Problem (1.2) has been investigated by Oskolkov and Kotsiolis [8], where the Ladyzhenskaja's methods were applied (see [7]). These investigations were continued in the articles of Agranovich and Sobolevskii [1, 2, 3], Sobolevskii [11, 12], Orlov and Sobolevskii [10] and Cannon et al. [4]. The existence and uniqueness of the solution of problem (1.2), local in time for $m = 3$ and global in time for $m = 2$, were established in [1, 2, 10]. The pair (u, p) is called the solution of problem (1.2) if their highest derivatives belong to $L^2([0,T]; L^2(\Omega))$ for some $T > 0$ (local results) or for arbitrary $T > 0$ (global results), and the equations and the initial-boundary conditions are satisfied weakly. A necessary condition for a such solution to exist is that $f(x,t) \in L^2([0,T]; L^2(\Omega)^2)$ and

$$u_0(x) \in W_0^2(\Omega)^2 = \big\{ v \in H^2(\Omega)^2 \cap H_0^1(\Omega)^2 \text{ with div } \ v = 0 \text{ in } \Omega \big\}.$$

An asymptotic series solution is constructed in [12] and the nonlinear Galerkin numerical method for the solution in the case of the periodic boundary condition is studied in [4].

The steady state flow $(\bar{u}, \bar{p})$ is a solution of the following boundary value problem. Find $(\bar{u}(x), \bar{p}(x))$ such that

$$\begin{cases} -(\epsilon + \dfrac{\rho}{\delta})\Delta\bar{u} + (\bar{u}\cdot\nabla)\bar{u} + \nabla\bar{p} = \bar{f}\,, \operatorname{div}\bar{u} = 0 \ (x\in\Omega)\,; \\ \bar{u} = 0 \ (x \in \partial\Omega); \ (\bar{p},1) = 0; \ \bar{f}(x) = \limsup\limits_{t\to\infty} f(x,t)\,. \end{cases} \tag{1.3}$$

Recently, the exponential convergence rate of $(u(x,t), p(x,t)$ to the steady state solution $(\bar{u}(x), \bar{p}(x))$ was shown by Sobelevskii [11]. Also, the convergence to the steady state in the case of the Navier-Stokes motion (or $\rho = 0$) in exterior domain was provided by Galdi et al. [5].

To prove the convergence of (u,p) to $(\bar{u},\bar{p})$, Sobolevskii [11] introduced the related self-adjoint spectral (or eigenvalue) problem:

$$\begin{cases} -\epsilon\Delta\bar{z} + \dfrac{1}{2}\big[(\dfrac{\partial\bar{u}}{\partial x}) + (\dfrac{\partial\bar{u}}{\partial x})^*\big]\bar{z} + \nabla\bar{r} = \lambda\bar{z}, \ \operatorname{div}\bar{z} = 0 \quad (x\in\Omega); \\ \bar{z} = 0 \ (x\in\partial\Omega); \ (\bar{r},1) = 0; \ \bar{z}\in W_0^2(\Omega)^2, \ \bar{r}\in H^1(\Omega), \end{cases} \tag{1.4}$$

where $(\bar{z},\bar{r})$ is the eigenfunction and λ the eigenvalue.

The exponential convergence can be described by the following theorem (Sobolevskii [11]).

Theorem 1.1. *Let $\rho \geq 0, \lambda_0 > 0$, $u_0(x) \in W_0^2(\Omega)^2$ and for some $\delta_1 \in (0, \min[\lambda_0, \delta])$, $\alpha \in (0,1), C_1 > 0$. If the function $\Phi(x,t) = e^{\delta_1 t}(f(x,t) - \bar{f}(x))$ satisfies*

$$\|\Phi(t)\|_{L^2(\Omega)^2} \leq C_1, \ \|\Phi(t) - \Phi(s)\|_{L^2(\Omega)^2} \leq C_1|t-s|^\alpha, \ 0 \leq s \leq t, \tag{1.5}$$

then for $m = 2$, the functions $z(x,t) = u(x,t) - \bar{u}(x)$ and $r(x,t) = p(x,t) - \bar{p}(x)$ satisfy

$$\|e^{\delta_1 t}z(t)\|_{H^2(\Omega)^2} + \|e^{\delta_1 t}z_t(t)\|_{L^2(\Omega)^2} + \|e^{\delta_1 t}r(t)\|_{H^1(\Omega)} \leq C_2, \tag{1.6}$$

for some $C_2 > 0$, where $z_t = \frac{\partial z}{\partial t}$ and λ_0 is the minimal eigenvalue of problem (1.4).

It was also shown in [11] that the estimates (1.6) are exact with respect to the rate of exponential convergence.

Remark 1.1. Since $\epsilon + \frac{\rho}{\delta} > 0$, then for an arbitrary vector-function $\bar{f}(x) \in L^2(\Omega)^2$, problem (1.3) has at least one solution $(\bar{u},\bar{p})$ such that $\bar{u}\in W_0^2(\Omega)^2, \bar{p}\in H^1(\Omega)/R$ satisfying

$$\|\bar{u}\|_{H^2(\Omega)^2} + \|\bar{p}\|_{H^1(\Omega)} \leq c\|\bar{f}\|_{L^2(\Omega)^2},$$

(see [6, 7, 13]).

Remark 1.2. If $(\epsilon + \frac{\rho}{\delta})$ and $\bar{f}$ satisfy the uniqueness condition:

$$\frac{N}{(\epsilon + \frac{\rho}{\delta})^2}\|\bar{f}\|_{L^2(\Omega)^2} < 1 \text{ and } N = c_0\lambda_1^{-1},$$

then $\lambda_0 > 0$ (see [6]), where $\lambda_1 > 0$ is the minimal eigenvalue of the Laplace operator $-\Delta$ and $c_0 > 0$ is a positive constant defined below in (2.3)-(2.4). $\lambda_0 > 0$ is also true for $\|\bar{u}\|_{C^1(\Omega)^2}$ small or other weaker conditions on f, see [11] for references.

In this article we shall mainly consider the exponential convergence and power convergence of $(u(x,t), p(x,t))$ to $(\bar{u}(x), \bar{p}(x))$ for two-dimensional viscoelastic fluid motion, where

$$\alpha \geq 0,\ \rho \geq 0,\ \lambda_1 > 0,\ 0 \leq \delta_1 < \delta_0 < \frac{1}{2}\min\{\delta, \nu\lambda_1\},$$

$$\alpha_0 = \delta - \delta_0, \text{ and } \alpha_1 = \delta_0 - \delta_1$$

are assumed, and where $\nu > 0$ will be defined in (2.6) in Section 2.

Our main results are included in the Theorems 1.2 - 1.4 below.

Theorem 1.2. *Let $u_0 \in H_0^1(\Omega)^2$ with div $u_0 = 0$ in Ω, $f \in L^2_{loc}(R^+; L^2(\Omega)^2)$ and $\bar{f} \in L^2(\Omega)^2$ satisfy for some $L \geq 0$,*

$$\limsup_{t\to\infty} \left(t^\alpha e^{\delta_1 t}\|f(t) - \bar{f}\|_{L^2(\Omega)^2}\right) = L < \infty, \tag{1.7}$$

then

$$\limsup_{t\to\infty} t^\alpha e^{\delta_1 t}\Big[\|z(t)\|_{H^2(\Omega)^2} + \|z_t(t)\|_{L^2(\Omega)^2} + \|r(t)\|_{H^1(\Omega)}\Big] \leq C_2 L, \tag{1.8}$$

holds for some $C_2 > 0$.

Theorem 1.3. *Let $u_0 \in H_0^1(\Omega)^2$ with div $u_0 = 0$ in Ω, $f \in L^2_{loc}(R^+; L^2(\Omega)^2)$ and $\bar{f} \in L^2(\Omega)^2$ satisfy*

$$\tau^\alpha(t)e^{\delta_1 t}\|f(t) - \bar{f}\|_{L^2(\Omega)^2} \leq C_1,\ \forall t \geq 0, \tag{1.9}$$

then there exists a constant $C_2 > 0$, independent of t, such that

$$\begin{aligned}\tau^\alpha(t)e^{\delta_1 t}\|z(s)\|_{H^1(\Omega)^2} &+ \left(e^{-2\alpha_1 t}\int_0^t \tau^{2\alpha}(s)\|e^{\delta_0 s}z(s)\|^2_{H^2(\Omega)^2}ds\right)^{1/2} \\ &+ \left(e^{-2\alpha_1 t}\int_0^t \tau^{2\alpha}(s)\|e^{\delta_0 s}z_t(s)\|^2_{L^2(\Omega)^2}ds\right)^{1/2} \\ &+ \left(e^{-2\alpha_1 t}\int_0^t \tau^{2\alpha}(s)\|e^{\delta_0 s}r(s,x)\|^2_{H^1(\Omega)})ds\right)^{1/2} \leq C_2,\end{aligned} \tag{1.10}$$

where $\tau(t) = \max\{\bar{t}, t\}$, $\bar{t} = \max\left\{\frac{\alpha}{\delta-\delta_0}, \frac{8\alpha}{\nu\lambda_1}, \frac{8\alpha}{\epsilon\lambda_1}\right\}$, *if* $\alpha > 0$, *and* $\tau(t) \equiv 1$, *if* $\alpha = 0$.

It is obvious that

$$\tau^{\alpha}(t) \geq \tau^{\alpha}(0) > 0, \ \forall t \geq 0, \ \alpha \geq 0.$$

Theorem 1.4. *Let* $u_0 \in W_0^2(\Omega)^2$, $f(x,t) \in L^2_{loc}(R^+; L^2(\Omega)^2)$, $\bar{f} \in L^2(\Omega)^2$ *and* $f_t \in L^2_{loc}(R^+; H^{-1}(\Omega)^2)$, *and there exists* $C_1 > 0$ *such that*

$$\tau^{\alpha}(t)e^{\delta_1 t}\Big[\|f(t) - \bar{f}\|_{L^2(\Omega)^2} + \|f_t(t)\|_{H^{-1}(\Omega)^2}\Big] \leq C_1, \ \forall t \geq 0, \tag{1.11}$$

then there exists $C_2 > 0$ *such that*

$$\tau^{\alpha}(t)e^{\delta_1 t}\Big[\|z(t)\|_{H^2(\Omega)^2} + \|z_t(t)\|_{L^2(\Omega)^2} + \|r(t)\|_{H^1(\Omega)}\Big] \leq C_2, \ \forall t \geq 0. \tag{1.12}$$

Remark 1.3. A special external force may satisfy

$$\limsup_{t\to\infty} t^{\alpha}e^{\delta_1 t}\|f(t) - \bar{f}\|_{L^2(\Omega)^2} = 0.$$

Then Theorem 1.2 yields

$$\limsup_{t\to\infty} t^{\alpha}e^{\delta_1 t}\Big(\|z(t)\|_{H^2(\Omega)^2} + \|z_t(t)\|_{L^2(\Omega)^2} + \|r(t)\|_{L^2(\Omega)}\Big) = 0, \tag{1.13}$$

or

$$\|z(t)\|_{H^2(\Omega)^2} + \|z_t(t)\|_{L^2(\Omega)^2} + \|r(t)\|_{H^1(\Omega)} = o(t^{-\alpha}e^{-\delta_1 t}), \text{ as } t \to \infty. \tag{1.14}$$

If $\rho = 0$, the above convergence results (including Theorem 1.1) are degenerated to the case for the Navier-Stokes flow.

The main technical difficulty that has been overcome in this paper is the elimination of Holder continuity of the function $\Phi(x,t)$, which makes the power convergence possible. Our arguments work well for both exponential and power convergence under assumptions of the external force being exponential or power decay, respectively, with respect to time. Thus, Theorems 1.2–1.4 extend the earlier results summarized by Theorem 1.1.

Another main contribution is the relaxation of the initial data assumptions in Theorem 1.2. This implies that our techniques allow us to obtain more general results under a weaker smoothness assumption on data.

The main conclusions are proved by variational approach and by using various differential inequalities. Section 2 sets up the variational formulation of the governing equations. Section 3 describes preliminary tools for proofs, and the main proofs are given in Sections 4–6.

2. Formulation of the problem

The mathematical setting of problem (1.2) needs the following Hilbert spaces $X = H_0^1(\Omega)^2$, $Y = L^2(\Omega)^2$, $M = L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q dx = 0\}$. The spaces $L^2(\Omega)^m$, $m = 1, 2, 4$ are endowed with the L^2-scalar product and L^2-norm denoted by $(\cdot,\cdot)$ and $|\cdot|$. The space $H_0^1(\Omega)$ and X are equipped with their usual scalar product and norm

$$((u, v)) = (\nabla u, \nabla v), \ \|u\| = ((u, u))^{1/2}.$$

The subspaces V and H of X and Y are defined by $V = \{v \in X; \text{div } v = 0 \text{ in } \Omega\}$, $H = \{v \in Y; \text{div } v = 0 \text{ in } \Omega \text{ and } v \cdot n = 0 \text{ on } \partial\Omega\}$, where n is the outnormal vector of Ω.

The Laplace operator is

$$Au = -\Delta u \quad \forall u \in D(A) = H_0^1(\Omega)^2 \cap H^2(\Omega)^2,$$

and the bilinear operator is

$$B(u, v) = (u \cdot \nabla)v, \ \forall u, v \in X.$$

The continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X \times X$ and $X \times M$ are

$$a(u, v) = \epsilon((u, v)), \ \forall u, v \in X,$$

$$d(v, q) = -(v, \nabla p) = (q, \text{div}v), \ \forall v \in X, q \in M,$$

and the trilinear form $b(\cdot,\cdot,\cdot)$ is

$$b(u, v, w) =< B(u, v), w >_{X',X}, \ \forall u, v, w \in X.$$

The above a, b and d satisfy the following properties (see [6, 13]):

$$\gamma|q| \le \sup_{v \in X} \frac{d(v, q)}{\|v\|}, \ \forall q \in M, \tag{2.1}$$

$$b(u, v, w) = -b(u, w, v) \quad \forall u, v, w \in X\,, \tag{2.2}$$

$$|b(u, v, w)| \le c_0|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2}, \ \forall u, v, w \in X, \tag{2.3}$$

$$|b(u, v, w)| \le c_0|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w|, \ \forall u \in X, v \in D(A), w \in Y,$$

$$|b(v, u, w)| \le c_0|v|^{1/2}|Av|^{1/2}\|u\||w|, \ \forall u \in X, v \in D(A), w \in Y, \tag{2.4}$$

where $\gamma > 0$ and $c_0 > 0$ are positive constants. In the following we also use c and $c_j's$ to denote generic positive constants.

From $\lambda_0 > 0$ and (2.2) it follows that for an arbitrary $\bar{z} \in W_0^2(\Omega)^2, \bar{r} \in H^1(\Omega) \cap M$, the inequality

$$a(\bar{z}, \bar{z}) + b(\bar{z}, \bar{z}, \bar{z}) + b(\bar{u}, \bar{z}, \bar{z}) + b(\bar{z}, \bar{u}, \bar{z}) = a(\bar{z}, \bar{z}) + b(\bar{z}, \bar{u}, \bar{z})$$

$$= \Big(\epsilon A\bar{z} + \frac{1}{2}[(\frac{\partial \bar{u}}{\partial x}) + (\frac{\partial \bar{u}}{\partial x})^*]\bar{z} + \nabla \bar{r}, \bar{z}\Big) \geq \lambda_0 |\bar{z}|^2 \tag{2.5}$$

holds. It follows from (2.5), Friedrichs inequality (see [7, 11]) and $a(\bar{z},\bar{z}) = \epsilon\|\bar{z}\|^2$, that there is $\nu > 0$ such that

$$a(\bar{z},\bar{z}) + b(\bar{z},\bar{z},\bar{z}) + b(\bar{u},\bar{z},\bar{z}) + b(\bar{z},\bar{u},\bar{z}) \geq \nu\|\bar{z}\|^2. \tag{2.6}$$

The equation (1.3) implies the steady state solution $(\bar{u},\bar{p})$ in (1.3) satisfies

$$\begin{cases} \bar{u}_t + \epsilon A\bar{u} + B(\bar{u},\bar{u}) + \rho\displaystyle\int_0^t e^{-\delta(t-s)}A\bar{u}ds + \nabla\bar{p} \\ \ = \bar{f}(x) - \dfrac{\rho}{\delta}e^{-\delta t}A\bar{u}(x) \ (t \geq 0, x \in \Omega), \\ \text{div}\bar{u} = 0 \ (t \geq 0,\ x \in \Omega); \\ \bar{u} = 0 \ (t \geq 0,\ x \in \partial\Omega); \\ \bar{u}(x,0) = \bar{u}(x)\,(x \in \Omega);\ (\bar{p},1) = 0. \end{cases} \tag{2.7}$$

The equations (1.2) and (2.7) imply that $(z,r) = (u-\bar{u}, p-\bar{p})$ satisfies

$$\begin{cases} z_t + \epsilon Az + B(z,z) + B(z,\bar{u}) + B(\bar{u},z) + \rho\displaystyle\int_0^t e^{-\delta(t-s)}Azds \\ \ +\nabla r = F(x,t), \\ \text{div}z = 0 \ (t \geq 0, x \in \Omega); \\ z = 0 \ (t \geq 0, x \in \partial\Omega); \\ z(x,0) = z_0 \quad (x \in \Omega);\ (r,1) = 0. \end{cases} \tag{2.8}$$

These lead to the following variational formulation for (z,r):

$$\begin{aligned} (z_t,v) \quad &+ \quad a(z,v) + b(z,z,v) + b(\bar{u},z,v) + b(z,\bar{u},v) + J(t;z,v) \\ &- \quad d(v,r) + d(z,q) = (F,v), \forall(v,q) \in (X,M), \end{aligned} \tag{2.9}$$

$$z(x,0) \quad = \quad z_0 = u_0(x) - \bar{u}(x) \in W_0^2(\Omega)^2, \tag{2.10}$$

where

$$J(t;z,v) = \rho e^{-\delta t}\Big(\int_0^t e^{\delta\tau}Az(\tau)d\tau, v\Big) = \rho e^{-\delta t}\Big(\Big(\int_0^t e^{\delta\tau}z(\tau)d\tau, v\Big)\Big),$$

$$F(x,t) = f(x,t) - \bar{f}(x) + \frac{\rho}{\delta}e^{-\delta t}A\bar{u}(x). \tag{2.11}$$

According to the assumptions on $f(t)$ and $\bar{f}$ in §1, there exists a positive constant $\kappa_\alpha > 0$, independent of t, such that the function $F(x,t)$ defined above $F(x,t) \in L^2_{loc}(R^+;Y)$, $F_t(x,t) \in L^2_{loc}(R^+;X')$ satisfies

$$\limsup_{t\to\infty} \tau^\alpha(t)e^{\delta_1 t}|F(t)| = \limsup_{t\to\infty} \tau^\alpha(t)e^{\delta_1 t}|f(t) - \bar{f}| \tag{2.12}$$

under the assumptions of Theorem 1.2, or satisfies

$$\frac{\tau^\alpha(t)}{\sqrt{\nu\lambda_1\alpha_1}}e^{\delta_1 t}|F(t)| \le \kappa_\alpha,\ \forall t \ge 0 \tag{2.13}$$

under the assumptions of Theorem 1.3, or satisfies

$$\frac{\tau^\alpha(t)}{\sqrt{\nu\lambda_1\alpha_1}}e^{\delta_1 t}|F(t)| + \frac{\sqrt{2}\tau^\alpha(t)}{\sqrt{\nu\alpha_1}}e^{\delta_1 t}\|F_t(t)\|_{-1} \le \kappa_\alpha,\ \forall t \ge 0 \tag{2.14}$$

under the assumptions of Theorem 1.4, where

$$\|F_t(t)\|_{-1} = \sup_{0\neq v\in X}\frac{(F_t(t),v)}{\|v\|}.$$

3. Preliminaries

This section is devoted to deriving several basic estimates of the solution (z,r) of problem (2.8). Hereafter, we always assume that

$$0 < \delta_0 < \tfrac{1}{2}\min\left\{\delta,\nu\lambda_1,\epsilon\lambda_1\right\},\ \alpha_0 = \delta - \delta_0 > 0,\ \beta = 2\alpha \ge 0. \tag{3.1}$$

Moreover, we can derive easily

$$\lambda_1|v|^2 \le \|v\|^2,\ \forall v \in X, \tag{3.2}$$

where $\lambda_1 > 0$ is the first eigenvalue of (1.4).

Lemma 3.1. *Assume that $u \in L^2_{loc}(R^+;X)$ and $t_0 \ge 0$. Then, for all $t \ge t_0$ and $0 \le \bar\delta < \delta$,*

$$\begin{aligned}&\int_{t_0}^t \tau^\beta(s)\ J(s;u,e^{2\bar\delta s}u(s))ds \qquad (3.3)\\ =&\frac{1}{2}\rho\tau^\beta(t)e^{-2(\delta-\bar\delta)t}\|\int_0^t e^{\delta\tau}u(\tau)d\tau\|^2 - \frac{1}{2}\rho\tau^\beta(t_0)e^{-2(\delta-\bar\delta)t_0}\|\int_0^{t_0} e^{\delta\tau}u(\tau)d\tau\|^2\\ &+\frac{\rho}{2}\int_{t_0}^t\Big(2(\delta-\bar\delta)\tau^\beta(s) - \frac{d}{ds}\tau^\beta(s)\Big)e^{-2(\delta-\bar\delta)s}\|\int_0^s e^{\delta\tau}u(\tau)d\tau\|^2 ds.\end{aligned}$$

Moreover, if $u \in L^2_{loc}(R^+;D(A))$, then for all $t \ge t_0$ and $0 \le \bar\delta < \delta$,

$$\begin{aligned}&\int_{t_0}^t \tau^\beta(s)\ J(s;u,e^{2\bar\delta s}Au(s))ds \qquad (3.4)\\ =&\frac{1}{2}\rho\tau^\beta(t)e^{-2(\delta-\bar\delta)t}|\int_0^t e^{\delta\tau}Au(\tau)d\tau|^2 - \frac{1}{2}\rho\tau^\beta(t_0)e^{-2(\delta-\bar\delta)t_0}|\int_0^{t_0} e^{\delta\tau}Au(\tau)d\tau|^2\\ &+\frac{1}{2}\rho\int_{t_0}^t\Big(2(\delta-\bar\delta)\tau^\beta(s) - \frac{d}{ds}\tau^\beta(s)\Big)e^{-2(\delta-\bar\delta)s}|\int_0^s e^{\delta\tau}Au(\tau)d\tau|^2 ds.\end{aligned}$$

Proof. The proofs are carried out by integration by parts in a straight forward manner and are hence omitted here. $\square$

Lemma 3.2. *Under the assumptions of Theorem* 1.2, *the solution* (z, r) *of the system* (2.8) *satisfies the following estimates: For* $\forall t \geq 0, \beta \geq 0$, *the following holds*

$$\begin{aligned}\tau^{\beta}(t)|e^{\delta_0 t}z(t)|^2 + \frac{\nu}{4}\int_0^t \tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2 ds + \rho\tau^{\beta}(t)e^{-2\alpha_0 t}\|\int_0^t e^{\delta\tau}z(\tau)d\tau\|^2 \\ + \rho\int_0^t \Big(2\alpha_0\tau^{\beta}(s) - \frac{d}{ds}\tau^{\beta}(s)\Big)e^{-2\alpha_0 s}\|\int_0^s e^{\delta\tau}z(\tau)d\tau\|^2 ds \\ \leq \tau^{\beta}(0)|z_0|^2 + G_{\beta}(t), \end{aligned} \tag{3.5}$$

where

$$G_{\beta}(t) = \frac{2}{\nu\lambda_1}\int_0^t \tau^{\beta}(s)|e^{\delta_0 s}F(s)|^2 ds.$$

Moreover, the estimate

$$\begin{aligned}\limsup_{t\to\infty}\tau^{\beta}(t)|e^{\delta_1 t}z(t)|^2 + \frac{\nu}{8\alpha_1}\limsup_{t\to\infty}\tau^{\beta}(t)\|e^{\delta_1 t}z(t)\|^2 \\ \leq \frac{1}{\nu\lambda_1\alpha_1}\limsup_{t\to\infty}\tau^{\beta}(t)|e^{\delta_1 t}F(t)|^2, \end{aligned} \tag{3.6}$$

holds.

Proof. Take $(v, q) = e^{2\delta_0 t}(z(t), r(t))$ in (2.9). Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|e^{\delta_0 t}z|^2 + a(z,z) + b(z,z,z) + b(\bar{u},z,z) + b(z,\bar{u},z) \\ + J(t; z, e^{2\delta_0 t}z(t)) = \delta_0|e^{\delta_0 t}z|^2 + (e^{\delta_0 t}F, e^{\delta_0 t}z)\,. \end{aligned} \tag{3.7}$$

From (3.1)-(3.2) and (2.6), one has

$$(e^{\delta_0 t}F, e^{\delta_0 t}z) \leq \lambda_1^{-1/2}|e^{\delta_0 t}F|\|e^{\delta_0 t}z\| \leq \frac{\nu}{4}\|e^{\delta_0 t}z\|^2 + \frac{1}{\nu\lambda_1}|e^{\delta_0 t}F|^2,$$

$$\nu\|z\|^2 \leq a(z,z) + b(z,z,z) + b(\bar{u},z,z) + b(z,\bar{u},z),$$

$$\delta_0|e^{\delta_0 t}z(t)|^2 \leq \frac{\nu\lambda_1}{2}|e^{\delta_0 t}z(t)|^2 \leq \frac{\nu}{2}\|e^{\delta_0 t}z(t)\|^2.$$

Hence, (3.7) impels that for all $t > 0$

$$\frac{d}{dt}|e^{\delta_0 t}z|^2 + \frac{\nu}{2}\|e^{\delta_0 t}z\|^2 + 2J(t; z, e^{2\delta_0 t}z(t)) \leq \frac{2}{\nu\lambda_1}|e^{\delta_0 t}F|^2, \tag{3.8}$$

which in turn implies that

$$\frac{d}{dt}(\tau^\beta(t)|e^{\delta_0 t}z|^2) - \frac{d\tau^\beta(t)}{dt}|e^{\delta_0 t}z(t)|^2 + \frac{\nu}{2}\tau^\beta(t)\|e^{\delta_0 t}z\|^2 + 2\tau^\beta(t)J(t;z,e^{2\delta_0 t}z(t)) \le \frac{2}{\nu\lambda_1}\tau^\beta(t)|e^{\delta_0 t}F|^2 . \tag{3.9}$$

Let $\bar{t} \ge \frac{4\beta}{\nu\lambda_1}$, the following holds for $\tau^\beta(t)$:

$$\frac{d}{dt}\tau^\beta(t) = 0,\ \forall t \ge 0 \text{ for } \beta = 0;\ \frac{d}{dt}\tau^\beta(t) = 0,\ \forall 0 \le t < \bar{t} \text{ for } \beta > 0;$$
$$\frac{d}{dt}\tau^\beta(t) = \beta\tau^{\beta-1}(t) \le \frac{\nu\lambda_1}{4}\bar{t}\tau^{\beta-1}(t) \le \frac{\nu\lambda_1}{4}\tau^\beta(t),\ t \in (\bar{t},\infty). \tag{3.10}$$

From (3.2) and (3.10) it follows that

$$\frac{d\tau^\beta(t)}{dt}|e^{\delta_0 t}z(t)|^2 \le \frac{\nu\lambda_1}{4}\tau^\beta(t)|e^{\delta_0 t}z(t)|^2 \le \frac{\nu}{4}\tau^\beta(t)\|e^{\delta_0 t}z(t)\|^2.$$

Hence, (3.9) yields that for all $t > 0$,

$$\frac{d}{dt}(\tau^\beta(t)|e^{\delta_0 t}z(t)|^2) + \frac{\nu}{4}\tau^\beta(t)\|e^{\delta_0 t}z(t)\|^2 + 2\tau^\beta(t)J(t;z,e^{2\delta_0 t}z(t)) \le \frac{2}{\nu\lambda_1}\tau^\beta(t)|e^{\delta_0 t}F(t)|^2. \tag{3.11}$$

Integrating (3.11) for t from 0 to t and using Lemma 3.1 with $\bar{\delta} = \delta_0$, we obtain that for all $t \ge 0$

$$\begin{aligned}\tau^\beta(t)|e^{\delta_0 t}z(t)|^2 &+ \frac{\nu}{4}\int_0^t \tau^\beta(s)\|e^{\delta_0 s}z(s)\|^2 ds + \rho\tau^\beta(t)e^{-2\alpha_0 t}\|\int_0^t e^{\delta\tau}z(\tau)d\tau\|^2 \\ &+ \rho\int_0^t \Big(2\alpha_0\tau^b(s) - \frac{d\tau^\beta(s)}{ds}\Big)e^{-2\alpha_0 s}\|\int_0^s e^{\delta\tau}z(\tau)d\tau\|^2 ds \\ &\le \tau^\beta(0)|z_0|^2 + G_\beta(t).\end{aligned} \tag{3.12}$$

Multiplying (3.5) by $e^{-2\alpha_1 t}$ and taking the limits for the resultant expression by L'Hopital rule as $t{\to}\infty$, we then obtain (3.6) since

$$\limsup_{t\to\infty} e^{-2\alpha_1 t}G_\beta(t) \quad = \quad \frac{1}{\nu\lambda_1\alpha_1}\limsup_{t\to\infty}\tau^\beta(t)|e^{\delta_1 t}F(t)|^2.$$

The proof is now complete. □

Lemma 3.3. *Under the assumptions of Theorem* 1.2, *the solution* (z, r) *of the system* (2.8) *satisfies the following for all* $\beta \geq 0, t \geq 0$,

$$\begin{aligned} &\tau^\beta(t)\|e^{\delta_0 t} z(t)\|^2 + \frac{\epsilon}{4}\int_0^t \tau^\beta(s)|e^{\delta_0 s} Az(s)|^2 ds + \rho\tau^\beta(t)e^{-2\alpha_0 t}\Big|\int_0^t e^{\delta\tau} Az(\tau)d\tau\Big|^2 \\ &+ \rho\int_0^t \Big(2\alpha_0\tau^\beta(s) - \frac{d}{ds}\tau^\beta(s)\Big)e^{-2\alpha_0 s}\Big|\int_0^s e^{\alpha\tau} Az(\tau)d\tau\Big|^2 ds \qquad (3.13) \\ &\leq (1+\eta)\tau^\beta(0)\|z_0\|^2 + (\eta + \frac{3\nu}{\epsilon})\lambda_1 G_\beta(t) + \int_0^t g(s)\tau^\beta(s)\|e^{\delta_0 s} z(s)\|^2 ds, \end{aligned}$$

where

$$\eta = \frac{96c_0^2}{\epsilon\nu\lambda_1^{3/2}}\|\bar{u}\||A\bar{u}| \quad \textit{and} \quad g(t) = 2\big(\frac{6}{\epsilon}\big)^3 c_0^4 |z(t)|^2\|z(t)\|^2.$$

Proof. From (2.8) it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|e^{\delta_0 t} z\|^2 + \epsilon|e^{\delta_0 t} Az|^2 + J(t; z, e^{2\delta_0 t} Az(t)) + b(z, e^{\delta_0 t} z, e^{\delta_0 t} z) \qquad (3.14) \\ &+ b(\bar{u}, e^{\delta_0 t} z, e^{\delta_0 t} Az) + b(e^{\delta_0 t} z, \bar{u}, e^{\delta_0 t} Az) = \delta_0\|e^{\delta_0 t} z\|^2 + (e^{\delta_0 t} F, e^{\delta_0 t} Az)\,. \end{aligned}$$

Then (3.1)-(3.2) and (2.4) imply that

$$\begin{aligned} &(e^{\delta_0 t} F, e^{\delta_0 t} z) \leq \frac{\epsilon}{12}|e^{\delta_0 t} Az|^2 + \frac{3}{\epsilon}|e^{\delta_0 t} F|^2, \\ &\frac{\epsilon}{2}|e^{\delta_0 t} Az(t)|^2 \geq \frac{\epsilon\lambda_1}{2}\|e^{\delta_0 t} z(t)\|^2 \geq \delta_0\|e^{\delta_0 t} z(t)\|^2, \\ &|b(\bar{u}, z, Az)| + |b(z, \bar{u}, Az)| \leq 2c_0\lambda_1^{-1/4}\|\bar{u}\|^{1/2}|A\bar{u}|^{1/2}\|z\||Az| \\ &\qquad \leq \frac{\epsilon}{12}|Az|^2 + \frac{12}{\epsilon\lambda_1^{1/2}}c_0^2\|\bar{u}\||A\bar{u}|^2\|z\|, \\ &|b(z, z, Az)| \leq c_0|z|^{1/2}\|z\||Az|^{3/2} \leq \frac{\epsilon}{12}|Az|^2 + (\frac{6}{\epsilon})^3 c_0^4|z|^2\|z\|^4. \end{aligned}$$

Hence, (3.14) becomes

$$\begin{aligned} &\frac{d}{dt}\|e^{\delta_0 t} z\|^2 + \frac{\epsilon}{2}|e^{\delta_0 t} Az|^2 + 2J(t; z, e^{2\delta_0 t} Az(t)) \\ &\qquad \leq \frac{\nu}{4}\lambda_1\eta\|e^{\delta_0 t} z(t)\|^2 + \frac{6}{\epsilon}|e^{\delta_0 t} F|^2 + g(t)\|e^{\delta_0 t} z(t)\|^2. \qquad (3.15) \end{aligned}$$

From (3.15) it follows that for $t > 0$,

$$\frac{d}{dt}(\tau^\beta(t)\|e^{\delta_0 t} z\|^2) - \frac{d\tau^\beta(t)}{dt}\|e^{\delta_0 t} z(t)\|^2 + \frac{\epsilon}{2}\tau^\beta(t)|e^{\delta_0 t} Az|^2$$

$$+ 2\tau^{\beta}(t)J(t; z, e^{2\delta_0 t}Az(t))$$
$$\leq \frac{\nu}{4}\lambda_1\eta\tau^{\beta}(t)\|e^{\delta_0 t}z(t)\|^2 + \frac{4}{\epsilon}\tau^{\beta}(t)|e^{\delta_0 t}F|^2 + \int_0^t g(s)\|e^{\delta_0 s}z(s)\|^2 ds\,. \quad (3.16)$$

Let $\bar{t} \geq \frac{4\beta}{\epsilon\lambda_1}$ and $\beta > 0$, then it follows from a similar estimate to (3.10):

$$\frac{d}{dt}\tau^{\beta}(t) \leq \frac{\epsilon\lambda_1}{4}\tau^{\beta}(t),\ \forall t \geq 0,\ \beta \geq 0. \quad (3.17)$$

Equations (3.2) and (3.17) lead to that for all $t > 0$

$$\frac{d\tau^{\beta}(t)}{dt}\|e^{\delta_0 t}z(t)\|^2 \leq \frac{\epsilon\lambda_1}{4}\tau^{\beta}(t)\|e^{\delta_0 t}z(t)\|^2 \leq \frac{\epsilon}{4}\tau^{\beta}(t)|e^{\delta_0 t}Az(t)|^2. \quad (3.18)$$

Hence, (3.16) and (3.18) yield

$$\frac{d}{dt}(\tau^{\beta}(t)\|e^{\delta_0 t}z(t)\|^2) + \frac{\epsilon}{4}\tau^{\beta}(t)|e^{\delta_0 t}Az(t)|^2 + 2\tau^{\beta}(t)J(t; z, e^{2\delta_0 t}Az(t))$$
$$\leq \frac{\nu}{4}\lambda_1\eta\tau^{\beta}(t)\|e^{\delta_0 t}z(t)\|^2 + \frac{4}{\epsilon}\tau^{\beta}(t)|e^{\delta_0 t}F(t)|^2 + g(t)\tau^{\beta}(t)\|e^{\delta_0 t}z(t)\|^2. \quad (3.19)$$

Integrating (3.19) for t from 0 to t and using Lemma 3.2, we obtain that for all $t \geq 0$

$$\tau^{\beta}(t)\|e^{\delta_0 t}z(t)\|^2 + \frac{\epsilon}{4}\int_0^t \tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2 ds$$
$$+ \rho\tau^{\beta}(t)e^{-2\alpha_0 t}|\int_0^t e^{\delta\tau}Az(\tau)d\tau|^2 + \rho\int_0^t 2\alpha_0 e^{-2\alpha_0 s}|\int_0^s e^{\delta\tau}A(\tau)d\tau|^2 ds$$
$$\leq \tau^{\beta}(0)\|z_0\|^2 + \frac{\nu}{4}\lambda_1\eta\int_0^t \tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2 ds \quad (3.20)$$
$$+ \frac{6}{\epsilon}\int_0^t \tau^{\beta}(s)|e^{\delta_0 s}F(s)|^2 ds + \int_0^t g(s)\tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2 ds\,.$$

Applying Lemma 3.2, one obtains

$$\frac{\nu}{4}\lambda_1\eta\int_0^t \tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2 ds + \frac{6}{\epsilon}\int_0^t \tau^{\beta}(s)|e^{\delta_0 s}F(s)|^2 ds$$
$$\leq \eta\tau^{\beta}(0)\|z_0\|^2 + (\eta + \frac{3\nu}{\epsilon})\lambda_1 G_{\beta}(t), \quad (3.21)$$

which together with (3.20) implies (3.13). □

4. Proof of Theorem 1.2

The variational formulation (2.9)-(2.10) implies

$$
\begin{aligned}
&|e^{\delta_0 t} z_t|^2 + \frac{\nu}{2}\frac{d}{dt}\|e^{\delta_0 t} z\|^2 + b(\bar{u}, e^{\delta_0 t} z, e^{\delta_0 t} z_t) + b(e^{\delta_0 t} z, \bar{u}, e^{\delta_0 t} z_t) \qquad (4.1)\\
&+ b(z, e^{\delta_0 t} z, e^{\delta_0 t} z_t) + J(t; z, e^{2\delta_0 t} z_t(t)) \le \delta_0 \|e^{\delta_0 t} z\|^2 + (e^{\delta_0 t} F, e^{\delta_0 t} z_t),
\end{aligned}
$$

and also from (2.4) and (3.1)-(3.2) it follows that

$$
\begin{aligned}
(e^{\delta_0 t} F, e^{\delta_0 t} z_t) &\le \frac{1}{6}|e^{\delta_0 t} z_t|^2 + \frac{3}{2}|e^{\delta_0 t} F|^2,\\
|b(\bar{u}, e^{\delta_0 t} z, e^{\delta_0 t} z_t)| &+ |b(e^{\delta_0 t} z, \bar{u}, e^{\delta_0 t} z_t)|\\
&\le 2c_0 \lambda_1^{-1/4} \|\bar{u}\|^{1/2} |A\bar{u}|^{1/2} \|e^{\delta_0 t} z\| |e^{\delta_0 t} z_t|^2\\
&\le \frac{1}{6}|e^{\delta_0 t} z_t(t)|^2 + 6c_0^2 \lambda_1^{-1/2} \|\bar{u}\| |A\bar{u}| \|e^{\delta_0 t} z\|^2\\
|b(z, e^{\delta_0 t} z, e^{\delta_0 t} z_t)| &\le c_0 e^{\delta_0 t} |z|^{1/2} \|z\| |Az|^{1/2} |e^{\delta_0 t} z_t|\\
&\le \frac{1}{6}|e^{\delta_0 t} z_t|^2 + \frac{3}{2} c_0^2 |z| \|z\| \|e^{\delta_0 t} z\| |e^{\delta_0 t} Az|.
\end{aligned}
$$

Hence, it follows from (4.1) that

$$
\begin{aligned}
&|e^{\delta_0 t} z_t|^2 + \nu \frac{d}{dt}\|e^{\delta_0 t} z\|^2 + 2e^{2\delta_0 t} J(t; z, z_t(t)) \qquad (4.2)\\
&\le 2(\delta_0 + 4c_0^2 \lambda_1^{-1/2} \|\bar{u}\| |A\bar{u}|) \|e^{\delta_0 t} z\|^2 + 2|e^{\delta_0 t} F|^2 + 3c_0^2 |z| \|z\| \|e^{\delta_0 t} z\| |e^{\delta_0 t} Az|.
\end{aligned}
$$

Multiplying (4.2) by $\tau^\beta(t)$ we find easily that

$$
\begin{aligned}
&\tau^\beta(t)|e^{\delta_0 t} z_t|^2 + \nu \frac{d}{dt}(\tau^\beta(t)\|e^{\delta_0 t} z\|^2) + 2\tau^\beta(t) e^{2\delta_0 t} J(t; z, z_t(t)) \qquad (4.3)\\
&\le \frac{d}{dt}\tau^\beta(t) \|e^{\delta_0 t} z(t)\|^2 + 2(\delta_0 + 4c_0^2 \lambda_1^{-1/2} \|\bar{u}\| |A\bar{u}|) \tau^\beta(t) \|e^{\delta_0 t} z\|^2\\
&\quad + 2\tau^\beta(t) |e^{\delta_0 t} F|^2 + 3c_0^2 |z| \|z\| \tau^\beta(t) \|e^{\delta_0 t} z\| |e^{\delta_0 t} Az|.
\end{aligned}
$$

Thus (3.10) and (4.3) yield that for all $\beta \ge 0$ and $t > 0$,

$$
\begin{aligned}
&\tau^\beta(t)|e^{\delta_0 t} z_t|^2 + \nu \frac{d}{dt}(\tau^\beta(t)\|e^{\delta_0 t} z\|^2) + 2\tau^\beta(t) e^{2\delta_0 t} J(t; z, z_t(t))\\
&\le (1 + \frac{8\delta_0}{\nu\lambda_1} + \epsilon\eta) \frac{\nu\lambda_1}{4} \tau^\beta(t) \|e^{\delta_0 t} z\|^2 \qquad (4.4)\\
&\quad + 3\tau^\beta(t) \Big[|e^{\delta_0 t} F|^2 + 3c_0^2 \tau^\beta(t) |z| \|z\| \|e^{\delta_0 t} z\| |e^{\delta_0 t} Az| \Big].
\end{aligned}
$$

Integration of (4.4) from 0 to t yields that for $\forall t \geq 0$,

$$\begin{aligned}
&\int_0^t \tau^\beta(s)|e^{\delta_0 s} z_t(s)|^2 ds + \nu\tau^\beta(t)\|e^{\delta_0 t} z(t)\|^2 + 2\int_0^t \tau^\beta(s) e^{2\delta_0 s} J(s; z, z_t(s)) ds \\
&\leq \nu\tau^\beta(0)\|z_0\|^2 + (1 + \frac{8\delta_0}{\nu\lambda_1} + \epsilon\eta)\frac{\nu\lambda_1}{4}\int_0^t \tau^\beta(s)\|e^{\delta_0 s} z(s)\|^2 ds \qquad (4.5) \\
&+ \frac{3}{2}\nu\lambda_1 G_\beta(t) + 3c_0^2\int_0^t |z|\|z\|\tau^\beta(s)\|e^{\delta_0 s} z\||e^{\delta_0 s} Az| ds.
\end{aligned}$$

The integration by parts leads to that

$$\begin{aligned}
&2\int_0^t \tau^\beta(s) e^{2\delta_0 s}\ J(s; z, z_t(s)) ds \\
&= 2\rho\int_0^t (\tau^\beta(s) e^{-(\delta - 2\delta_0)s}\int_0^s e^{\delta\tau} Az(\tau) d\tau, z_t(s)) ds \\
&= 2\rho\tau^\beta(t) e^{-(\delta-2\delta_0)t}\Big(\Big(\int_0^t e^{\delta\tau} z(\tau) d\tau, z(t)\Big)\Big) - 2\rho\int_0^t \tau^\beta(s)\|e^{\delta_0 s} z(s)\|^2 ds \\
&+ 2\rho\int_0^t \Big(2\alpha_0\tau^\beta(s) - \frac{d}{ds}\tau^\beta(s)\Big) e^{-\alpha_0 s}\Big(\Big(\int_0^s e^{\delta\tau} z(\tau) d\tau, e^{\delta_0 t} z(t)\Big)\Big) ds \qquad (4.6) \\
&- \rho\delta\int_0^t \tau^\beta(s) e^{-2\alpha_0 s}\frac{d}{ds}\|\int_0^s e^{\delta\tau} z(\tau) d\tau\|^2 ds = I_1(t) + I_2(t) + I_3(t) + I_4(t),
\end{aligned}$$

the Cauchy and Young inequalities, together with integration by parts, imply that

$$\begin{aligned}
|I_1(t)| &\leq 2\rho\tau^\beta(t) e^{-\alpha_0 t}\|\int_0^t e^{\delta\tau} z(\tau)\|\ \|e^{\delta_0 t} z(t)\| \\
&\leq \rho\tau^\beta(t) e^{-2\alpha_0 t}\|\int_0^t e^{\delta\tau} z(\tau)\|^2 + \rho\tau^\beta(t)\|e^{\delta_0 t} z(t)\|^2, \\
I_2(t) &= -2\rho\int_0^t \tau^\beta(s)\|e^{\delta_0 s} z(s)\|^2 ds, \\
|I_3(t)| &\leq 2\rho\Big(\int_0^t \Big(2\alpha_0\tau^\beta(s) - \frac{d}{ds}\tau^b(s)\Big) e^{-2\alpha_0 s}\|\int_0^s e^{\delta\tau} z(\tau) d\tau\|^2 ds\Big)^{1/2} \\
&\times\Big(\int_0^t \Big(2\alpha_0\tau^\beta(s) - \frac{d}{ds}\tau^\beta(s)\Big)\|e^{\delta_0 s} z(s)\|^2 ds\Big)^{1/2} \\
&\leq \rho\int_0^t \Big(2\alpha_0\tau^\beta(s) - \frac{d}{ds}\tau^\beta(s)\Big) e^{-2\alpha_0 s}\|\int_0^s e^{\delta\tau} z(\tau)\|^2 ds
\end{aligned}$$

$$
\begin{aligned}
&+ \rho\int_0^t \Big(2\alpha_0\tau^\beta(s) - \frac{d}{ds}\tau^\beta(s)\Big)\|e^{\delta_0 s}z(s)\|^2 ds\\
&\le \rho\int_0^t \Big(2\alpha_0\tau^\beta(s) - \frac{d}{ds}\tau^\beta(s)\Big)e^{-2\alpha_0 s}\|\int_0^s e^{\delta\tau}z(\tau)d\tau\|^2 ds\\
&+ 2\alpha_0\rho\int_0^t s^\beta\|e^{\delta_0 s}z(s)\|^2 ds,\\
I_4(t) &= -\rho\delta\int_0^t \tau^\beta(s)e^{-2\alpha_0 s}\frac{d}{ds}\|\int_0^s e^{\delta\tau}z(\tau)d\tau\|^2 ds\\
&= -\rho\delta\tau^\beta(t)e^{-2\alpha_0 t}\|\int_0^t e^{\delta\tau}z(\tau)d\tau\|^2\\
&- \rho\delta\int_0^t \Big(2\alpha_0\tau^\beta(s) - \frac{d}{ds}\tau^\beta(s)\Big)e^{-2\alpha_0 s}\|\int_0^s e^{\delta\tau}z(\tau)d\tau\|^2 ds.
\end{aligned}
$$

From the above inequalities it follows that

$$
\begin{aligned}
&2|\int_0^t \tau^\beta(s)J(s;z,e^{2\delta_0 s}z_t(s))ds| \le |I_1(t)| + |I_2(t) + |I_3(t)| + |I_4(t)|\\
&\le \rho\tau^\beta(t)\|e^{\delta_0 t}z(t)\|^2 + 2\rho(1+\alpha_0)\int_0^t \tau^\beta(s)\|e^{\delta_0 s}z(s)\|^2 ds\\
&+ \rho(1+\delta)\tau^\beta(t)e^{-2\alpha_0 t}\|\int_0^t e^{\delta\tau}z(\tau)d\tau\|^2\\
&+ \rho(1+\delta)\int_0^t \Big(2\alpha_0\tau^\beta(s) - \frac{d}{ds}\tau^\beta(s)\Big)e^{-2\alpha_0 s}\|\int_0^s e^{\delta\tau}z(\tau)d\tau\|^2 ds.
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
&2\int_0^t \tau^\beta(s)J(s;z,e^{2\delta_0 s}z_t(s))ds \le |I_1(t)| + |I_3(t)|\\
&\le \rho\tau^\beta(t)\|e^{\delta_0 t}z(t)\|^2 + 2\rho\alpha_0\int_0^t \tau^\beta(s)\|e^{\delta_0 s}z(s)\|^2 ds\\
&+ \rho\tau^\beta(t)e^{-2\alpha_0 t}\|\int_0^t e^{\delta\tau}z(\tau)d\tau\|^2\\
&+ \rho\delta\int_0^t \Big(2\alpha_0\tau^\beta(s) - \frac{d}{ds}\tau^\beta(s)\Big)e^{-2\alpha_0 s}\|\int_0^s e^{\delta\tau}z(\tau)d\tau\|^2 ds.
\end{aligned}
\tag{4.8}
$$

Combining (4.5) with (4.7) and using Lemma 3.2 and Lemma 3.3, it follows

$$
\int_0^t \tau^\beta(s)|e^{\delta_0 s}z_t(s)|^2 ds \le c\Big[\tau^\beta(t)\|e^{\delta_0 t}z(t)\|^2 + \int_0^t \tau^\beta(s)\|e^{\delta_0 s}z(s)\|^2 ds
$$

$$
\begin{aligned}
&+ \rho \int_0^t \Big(2\alpha_0 \tau^\beta(s) - \frac{d\tau^\beta(s)}{ds}\Big)\| \int_0^s e^{\delta\tau} z(\tau) d\tau\|^2 ds \\
&+ G_\beta(t) + \rho\tau^\beta(t) e^{-2\alpha_0 t}\| \int_0^t e^{\delta\tau} z(\tau) d\tau\|^2\Big] \\
&+ 3c_0^2 \int_0^t \tau^\beta(s)|z|\|z\|\|e^{\delta_0 s} z\||e^{\delta_0 s} Az| ds, \\
&\le c\Big[\tau^\beta(0)\|z_0\|^2 + G_b(t)\Big] + 3c_0^2 \int_0^t \tau^\beta(s)|z|\|z\|\|e^{\delta_0 s} z\||e^{\delta_0 s} Az| ds.
\end{aligned}
$$

This inequality and Lemma 3.3 yields $\forall t \ge 0$

$$
\begin{aligned}
&2\alpha_1 e^{-2\alpha_1 t}\Big[\int_0^t \tau^\beta(s)|e^{\delta_0 s} z_t(s)|^2 ds + \int_0^t \tau^\beta(s)|e^{\delta_0 s} Az(s)|^2 ds\Big] \\
&\le 2\alpha_1 c e^{-2\alpha_1 t}\Big[\tau^\beta(0)\|z_0\|^2 + G_\beta(t) + \int_0^t g(s)\tau^\beta(s)\|e^{\delta_0 s} z(s)\|^2 ds\Big].
\end{aligned} \tag{4.9}
$$

If the assumptions of Theorem 1.2 are true, we can take the limits for (4.9) as $t\to\infty$. By using (2.12) and Lemma 3.2 together with the L'Hopital's rule, we then derive that

$$
\begin{aligned}
&\limsup_{t\to\infty} \tau^\beta(t)|e^{\delta_1 t} z_t(t)|^2 + \limsup_{t\to\infty} \tau^\beta(t)|e^{\delta_1 t} Au(t)|^2 \\
&\le c\Big[\limsup_{t\to\infty} g(t)\tau^\beta(t)\|e^{\delta_1 t} z(t)\|^2 + \limsup_{t\to\infty} \tau^\beta(t)|e^{\delta_1 t} F(t)|^2\Big] \\
&\le c \limsup_{t\to\infty} \tau^\beta(t)|e^{\delta_1 t}(f(t) - \bar{f})|^2.
\end{aligned} \tag{4.10}
$$

Moreover, from (2.8) and (2.4) it follows that

$$
\begin{aligned}
&\tau^\beta(t)|e^{\delta_1 t}\nabla r(t)|^2 \le c\Big[\tau^\beta(t)|e^{\delta_1 t} z_t(t)|^2 + \tau^\beta(t)|e^{\delta_1 t} Az(t)|^2 \\
&+ \rho e^{-2(\delta-\delta_1)t}\tau^\beta(t)|\int_0^t e^{\delta\tau} Az(\tau)d\tau|^2 + g(t)\tau^\beta(t)\|e^{\delta_1 t} z(t)\|^2 + e^{-2\alpha_1 t} G_\beta(t)\Big].
\end{aligned} \tag{4.11}
$$

The application of (2.1) and (3.2) leads to

$$
\begin{aligned}
\gamma\tau^\alpha(t)|e^{\delta_1 t} r(t)| &\le \sup_{v\in X} \frac{d(v, \tau^\alpha(t) e^{\delta_1 t} r(t))}{\|v\|} \\
&= \sup_{v\in X} \frac{-\big(v, \tau^\alpha(t) e^{\delta_1 t}\nabla r(t)\big)}{\|v\|} \le \lambda_1^{-1/2}\tau^\alpha(t)|e^{\delta_1 t}\nabla r(t)|.
\end{aligned}
$$

This and (4.11) imply

$$\tau^{\beta}(t)\|e^{\delta_1 t}r(t)\|^2_{H^1(\Omega)} \le c\Big[(\tau^{\beta}(t)|e^{\delta_1 t}z_t(t)|^2 + \tau^{\beta}(t)|e^{\delta_1 t}Az(t)|^2 \tag{4.12}$$
$$+ \tau^{\beta}(t)e^{-2(\delta-\delta_1)t}|\int_0^t e^{\delta\tau}Az(\tau)d\tau|^2 + g(t)\tau^{\beta}(t)\|e^{\delta_1 t}z(t)\|^2 + e^{-2\alpha_1 t}G_{\beta}(t)\Big].$$

Next, by using Lemmas 3.2–3.3, (2.12) and (1.7), we find

$$\limsup_{t\to\infty} g(t)\tau^{\beta}(t)\|e^{\delta_1 t}z(t)\| \le c\limsup_{t\to\infty}\tau^{\beta}(t)\|e^{\delta_1 t}z(t)\|^2$$
$$\le c\limsup_{t\to\infty}\tau^{\beta}(t)|e^{\delta_1 t}F(t)|^2,$$
$$\limsup_{t\to\infty} e^{-2\alpha_1 t}\tau^{\beta}(t)e^{-2\alpha_0 t}|\int_0^t e^{\delta\tau}Az(\tau)d\tau|^2 \tag{4.13}$$
$$\le c\limsup_{t\to\infty} e^{-2\alpha_1 t}\Big(\tau^{\beta}(0)\|z_0\|^2 + G_{\beta}(t)\Big) \le c\limsup_{t\to\infty}\tau^{\beta}(t)|e^{\delta_1 t}F(t)|^2.$$

Hence, taking the limits for (4.11) as $t\to\infty$ and using (4.9) and (4.12), one can obtain

$$\limsup_{t\to\infty}\tau^{\beta}(t)\|e^{\delta_1 t}r(t)\|^2_{H^1(\Omega)})ds$$
$$\le c\Big[\limsup_{t\to\infty}\tau^{\beta}(t)|e^{\delta_1 t}z_t(t)|^2 + \limsup_{t\to\infty}\tau^{\beta}(t)|e^{\delta_1 t}Az(t)|^2$$
$$+\limsup_{t\to\infty} g(t)\tau^{\beta}(t)\|e^{\delta_1 t}z(t)\|^2 + \limsup_{t\to\infty}\tau^{\beta}(t)|e^{d_1 t}F(t)|^2\Big]$$
$$\le c\limsup_{t\to\infty}\tau^{\beta}(t)|e^{\delta_1 t}(f(t)-\bar{f})|^2. \tag{4.14}$$

Combining (4.10) and (4.14) completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

A similar argument to the ones in Section 3 yields the following estimates.

Lemma 5.1. *Under assumptions of Theorem* 1.3 *there hold for all* $t,\ \beta\ge 0$,

$$\tau^{\beta}(t)|e^{\delta_0 t}z(t)|^2 + \frac{\nu}{2}\int_0^t \tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2 ds + \rho\tau^{\beta}(t)e^{-2\alpha_0 t}\|\int_0^t e^{\delta\tau}z(\tau)d\tau\|^2$$
$$+\rho\int_0^t\big(2\alpha_0\tau^{\beta}(s) - \frac{d}{ds}\tau^{\beta}(s)\big)e^{-2\alpha_0 s}\|\int_0^s e^{\delta\tau}z(\tau)d\tau\|^2 ds \le \tau^{\beta}(0)|z_0|^2 + G_{\beta}(t), \tag{5.1}$$

$$
\begin{aligned}
&\tau^{\beta}(t)\|e^{\delta_0 t}z(t)\|^2+\frac{\epsilon}{4}\int_0^t\tau^{\beta}(s)|e^{\delta_0 s}Az(s)|^2ds+\rho\tau^{\beta}(t)e^{-2\alpha_0 t}|\int_0^t e^{\delta\tau}Az(\tau)d\tau|^2\\
&+\rho\int_0^t\Big(2\alpha_0\tau^{\beta}(s)-\frac{d}{ds}\tau^{\beta}(s)\Big)e^{-2\alpha_0 s}|\int_0^s e^{\alpha\tau}Az(\tau)d\tau|^2ds \qquad (5.2)\\
&\le(1+\eta)\tau^{\beta}(0)\|z_0\|^2+(\eta+\frac{3\nu}{\epsilon})\lambda_1G_{\beta}(t)+\int_0^t g(s)\tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2ds,
\end{aligned}
$$

$$
\frac{d}{dt}(\tau^{\beta}(t)|e^{\delta_0 t}z|^2)+\frac{\nu}{4}\tau^{\beta}(t)\|e^{\delta_0 t}z\|^2+2\tau^{\beta}(t)J(t;z,e^{2\delta_0 t}z(t))\le\frac{2}{\nu\lambda_1}\tau^{\beta}(t)|e^{\delta_0 t}F|^2 \tag{5.3}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}(\tau^{\beta}(t)\|e^{\delta_1 t}z(t)\|^2)+\frac{\epsilon}{4}\tau^{\beta}(t)|e^{\delta_1 t}Az(t)|^2+2\tau^{\beta}(t)J(t;z,e^{2\delta_1 t}Az(t))\\
&\le\frac{\nu}{4}\lambda_1\eta\tau^{\beta}(t)\|e^{\delta_1 t}z(t)\|^2+\frac{4}{\epsilon}\tau^{\beta}(t)|e^{\delta_1 t}F(t)|^2+g(t)\tau^{\beta}(t)\|e^{\delta_1 t}z(t)\|^2. \qquad (5.4)
\end{aligned}
$$

Proof. The proof of this lemma is similar to that of Lemmas 3.1–3.3 and is hence omitted. □

Next, The following estimates needs to be established: There exists $c_2>0$, independent of t, such that

$$
\tau^{\beta}(t)\|e^{\delta_1 t}z(t)\|^2\le c_2,\ \forall t,\beta\ge 0. \tag{5.5}
$$

From (5.1) and (2.13) it follows that

$$
\tau^{\beta}(t)|e^{\delta_1 t}z(t)|^2\le\tau^{\beta}(0)|z_0|^2+e^{-2\alpha_1 t}G_{\beta}(t)\le\tau^{\beta}(0)|z_0|^2+\kappa_{\alpha}^2,\forall t\ge 0, \tag{5.6}
$$

since

$$
e^{-2\alpha_1 t}G_{\beta}(t)\le\frac{2}{\nu\lambda_1}e^{-2\alpha_1 t}\int_0^t\tau^{\beta}(s)|e^{\delta_0 s}F(s)|^2ds\le\kappa_a^2. \tag{5.7}
$$

It follows from (5.3) that

$$
\frac{d}{dt}|e^{\delta_1 t}z|^2\quad+\quad\frac{\nu}{4}\|e^{\delta_1 t}z\|^2+2J(t;z,e^{2\delta_1 t}z(t))\le\frac{2}{\nu\lambda_1}|e^{\delta_1 t}F|^2. \tag{5.8}
$$

Integrating (5.8) from 0 to t and using Lemma 3.1 with $t_0=0,\beta=0$ and $\bar{\delta}=\delta_1$ imply

$$
\frac{\nu}{4}\int_0^t\|e^{\delta_1 s}z(s)\|^2ds\le|z_0|^2+\frac{2}{\nu\lambda_1}\int_0^t|e^{\delta_1 s}F(s)|^2ds\le|z_0|^2+2\alpha_1\kappa_0^2t,\ \forall t\ge 0. \tag{5.9}
$$

Combination of (5.9) and (5.6) yields

$$\int_0^t g(s)ds \le 2(\frac{6}{\epsilon})^3 c_0^4 \int_0^t |z(s)|^2 \|z(s)\|^2 ds \le c_3(1+t), \forall t \ge 0. \tag{5.10}$$

Let

$$y(t)=\tau^\beta(t)\|e^{\delta_1 t}z(t)\|^2,\ h(t)=(\frac{2\eta}{\nu}+\frac{6}{\epsilon})\tau^\beta(t)|e^{\delta_1 t}F(t)|^2,\ C=(1+\eta)\tau^\beta(0)\|z_0\|^2.$$

We have from (5.2) that

$$y(t) \le C + \int_0^t h(s)ds + \int_0^t g(s)y(s)ds, \forall t \ge 0. \tag{5.11}$$

The general Gronwall lemma (see [4]) is applied to (5.11) to obtain

$$\begin{aligned}\tau^\beta(t)\|e^{\delta_0 t}z(t)\|^2 = y(t) &\le \Big(C + \int_0^t h(s)ds\Big)\exp\int_0^t g(s)ds \qquad (5.12)\\ &\le \Big(C + (\frac{2\eta}{\nu}+\frac{6}{\epsilon})\int_0^t \tau^\beta(s)|e^{\delta_1 s}F(s)|^2 ds\Big)\exp(\int_0^t g(s)ds)\\ &\le \Big(C + (\frac{2\eta}{\nu}+\frac{6}{\epsilon})\int_0^t \tau^\beta(s)|e^{\delta_1 s}F(s)|^2 ds\Big)\exp(c_3(1+t)),\ \forall t \ge 0.\end{aligned}$$

From Lemma 3.2 and (1.9), we have

$$\limsup_{t\to\infty} \tau^\beta(t)\|e^{\delta_1 t}z(t)\|^2 \le \frac{8}{\nu^2\lambda_1} \limsup_{t\to\infty} \tau^\beta(t)|e^{\delta_1 t}F(t)|^2 \le \frac{8\alpha_1}{\nu}\kappa_\alpha^2. \tag{5.13}$$

Hence, there exists a finite time $T > 0$ such that

$$\tau^\beta(t)\|e^{\delta_1 t}z(t)\|^2 \le \frac{16\alpha_1}{\nu}\kappa_a^2, \quad \forall t \ge T. \tag{5.14}$$

Combining (5.12) with $0 \le t \le T$ and (5.14) give (5.5).

From (5.5)-(5.7), we derive from (4.9) and Lemma 3.2 that

$$\begin{aligned}&2\alpha_1 e^{-2\alpha_1 t}\Big[\int_0^t \tau^\beta(s)|e^{\delta_0 s}z_t(s)|^2 ds + \int_0^t \tau^\beta(s)|e^{\delta_0 s}Az(s)|^2 ds\Big] \qquad (5.15)\\ &\le 2\alpha_1 c\Big[e^{-2\alpha_1 t}[\tau^\beta(0)\|z_0\|^2 + G_\beta(t) + \int_0^t \tau^\beta(s)\|e^{\delta_0 s}z(s)\|^2 ds\Big] \le c_5,\ \forall t \ge 0.\end{aligned}$$

Moreover, from (2.8) and (2.4) it follows that

$$\begin{aligned}&\tau^\beta(t)|e^{\delta_0 t}\nabla r(t)|^2 \le c\Big[\tau^\beta(t)|e^{\delta_0 t}z_t(t)|^2 + \tau^\beta(t)|e^{\delta_0 t}Az(t)|^2\\ &+ \rho e^{-2\alpha_0 t}\tau^\beta(t)|\int_0^t e^{\delta\tau}Az(\tau)d\tau|^2 + g(t)\tau^\beta(t)\|e^{\delta_0 t}z(t)\|^2 + G_\beta(t)\Big]. \qquad (5.16)\end{aligned}$$

Again using (2.1) and (3.2), we obtain

$$\gamma\tau^{\alpha}(t)|e^{\delta_1 t}r(t)| \le \sup_{v\in X}\frac{d(v,\tau^{\alpha}(t)e^{\delta_1 t}r(t))}{\|v\|}$$
$$= \sup_{v\in X}\frac{-(v,\tau^{\alpha}(t)e^{\delta_1 t}\nabla r(t))}{\|v\|} \le \lambda_1^{-1/2}|\tau^{\alpha}(t)e^{\delta_1 t}\nabla r(t)|,$$

which together with (5.16) gives

$$\tau^{\beta}(t)\|e^{\delta_0 t}r(t)\|^2_{H^1(\Omega)} \le c\Big[(\tau^{\beta}(t)|e^{\delta_0 t}z_t(t)|^2 + \tau^{\beta}(t)|e^{\delta_0 t}Az(t)|^2$$
$$+ \tau^{\beta}(t)e^{-2\alpha_0 t}|\int_0^t e^{\delta\tau}Az(\tau)d\tau|^2 + g(t)\tau^{\beta}(t)\|e^{\delta_0 t}z(t)\|^2 + G_{\beta}(t)\Big]. \quad (5.17)$$

By using (5.5)-(5.6) and Lemma 3.2, we find

$$\tau^{\beta}(t)e^{-2\alpha_0 t}|\int_0^t e^{\delta\tau}Az(\tau)d\tau|^2$$
$$\le c\Big[\tau^{\beta}(0)\|z_0\|^2 + G_{\beta}(t) + \int_0^t g(s)\tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2 ds\Big]$$
$$\le c\Big[\tau^{\beta}(0)\|z_0\|^2 + G_{\beta}(t) + \int_0^t \tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2 ds\Big]. \quad (5.18)$$

Combining (5.17) with (5.18) and using (5.7) give

$$\tau^{\beta}(t)\|e^{\delta_0 t}r(t)\|^2_{H^1(\Omega)} \le c\Big[(\tau^{\beta}(t)|e^{\delta_0 t}z_t(t)|^2 + \tau^{\beta}(t)|e^{\delta_0 t}Az(t)|^2$$
$$+ \|z_0\|^2 + e^{2\alpha_1 t}\kappa_{\alpha}^2 + g(t)\tau^{\beta}(t)\|e^{\delta_0 t}z(t)\|^2\Big]. \quad (5.19)$$

Integrating (5.19) and using (5.15), we obtain

$$\int_0^t \tau^{\beta}(s)\|e^{\delta_0 s}r(s)\|^2_{H^1(\Omega)}ds \le c\Big[e^{2\alpha_1 t} + \int_0^t g(s)\tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2 ds\Big].$$

This and (5.5) lead to

$$\int_0^t \tau^{\beta}(s)\|e^{\delta_0 s}r(s)\|^2_{H^1(\Omega)}ds \le c\Big[e^{2\alpha_1 t} + \int_0^t \tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2 ds\Big]. \quad (5.20)$$

From Lemma 3.2 and (5.20) it follows that

$$e^{-2\alpha_1 t}\int_0^t \tau^{\beta}(s)\|e^{\delta_0 s}r(s)\|^2_{H^1(\Omega)}ds \le c_6, \forall t \ge 0. \quad (5.21)$$

From (5.5), (5.15) and (5.21), Theorem 1.3 follows.

6. Proof of Theorem 1.4

The proof of Theorem 1.4 needs the following results.

Theorem 6.1. *Under the assumptions of Theorem* 1.4, *the solution* (z, r) *of the system* (2.8) *satisfies the following estimates:*

$$\tau^{\beta}(t)|e^{\delta_1 t} z_t(t)|^2 \leq c_7, \quad \forall t \geq 0, \tag{6.1}$$

for some $c_7 > 0$.

Proof. Recalling (2.9)-(2.10), we can obtain

$$\begin{aligned}(z_{tt}, v) + (1 + \frac{\rho}{\epsilon})a(z_t, v) &+ b(\bar{u}, z_t, v) + b(z_t, \bar{u}, v)\\ &+ b(z_t, z, v) + b(z, z_t, v) - d(v, r_t) + d(z_t, q) \qquad (6.2)\\ &= (F_t, v) + \delta J(t; Az, v)\forall(v, q) \in (X, M),\\ z_t(0, x) &= \lim_{t\to 0} z_t(x, t) = z_{t0}(x), \qquad (6.3)\end{aligned}$$

where

$$|z_{t0}| \leq (\epsilon + \frac{\rho}{\delta})|Az_0| + 2c_0\lambda_1^{-1/2}\|\bar{u}\||Az_0| + |f(0, x) - \bar{f}(x)| + \frac{\rho}{\delta}|A\bar{u}(x)|.$$

Taking $(v, q) = e^{2\delta_0 t}(z_t, r_t)$ in (6.2) and using (2.2) and (2.6), we then find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|e^{\delta_0 t}z_t|^2 &+ (\nu + \rho)\|e^{\delta_0 t}z_t\|^2 + b(e^{\delta_0 t}z_t, z, e^{\delta_0 t}z_t)\\ &= \delta J(t; z, e^{2\delta_0 t}z_t(t)) + \delta_0|e^{\delta_0 t}z_t|^2 + (e^{\delta_0 t}F_t, e^{\delta_0 t}z_t) \qquad (6.4)\end{aligned}$$

From (3.1)-(3.2) and (2.3), one can derive

$$\begin{aligned}(e^{\delta_0 t}F_t, e^{\delta_0 t}z_t) &\leq \frac{\nu}{8}\|e^{\delta_0 t}z_t\|^2 + \frac{2}{\epsilon}\|e^{\delta_0 t}F_t\|_{-1}^2,\\ \frac{\nu}{2}\|e^{\delta_0 t}z_t(t)\|^2 &\geq \frac{\nu\lambda_1}{2}|e^{\delta_0 t}z_t(t)|^2 \geq \delta_0|e^{\delta_0 t}z_t(t)|^2,\\ |b(e^{\delta_0 t}z_t, z, e^{\delta_0 t}z_t)| &\leq \frac{\nu}{8}\|e^{\delta_0 t}z_t\|^2 + \frac{2}{\nu}c_0^2\|z\|^2|e^{\delta_0 t}z_t|^2.\end{aligned}$$

Hence, (6.4) and the above inequalities imply that

$$\begin{aligned}\frac{d}{dt}|e^{\delta_0 t}z_t|^2 &+ (\rho + \frac{\nu}{2})\|e^{\delta_0 t}z_t\|^2 \qquad (6.5)\\ &\leq 2\delta e^{2\delta_0 t}J(t; z, z_t(t)) + \frac{4}{\nu}\|e^{\delta_0 t}F_t\|_{-1}^2 + \frac{4}{\nu}c_0^2\|z\|^2|e^{\delta_0 t}z_t|^2, \ \forall t \geq 0.\end{aligned}$$

From (6.5) it follows that for all $\beta \geq 0$ and $t > 0$,

$$\frac{d}{dt}(\tau^\beta(t)|e^{\delta_0 t}z_t|^2) - \frac{d}{dt}\tau^\beta(t)|e^{\delta_0 t}z_t(t)|^2 + (\rho + \frac{\nu}{2})\tau^\beta(t)\|e^{\delta_0 t}z_t\|^2 \tag{6.6}$$
$$\leq \delta\tau^\beta(t)e^{2\delta_0 t}J(t;z,z_t(t)) + \frac{4}{\nu}\tau^\beta(t)\|e^{\delta_0 t}F_t\|_{-1}^2 + \frac{4}{\nu}c_0^2\tau^\beta(t)\|z\|^2|e^{\delta_0 t}z_t|^2.$$

By using (3.10) and (3.2), we have

$$\frac{d}{dt}\tau^\beta(t)|e^{\delta_0 t}z_t|^2 \leq \frac{\nu\lambda_1}{4}\tau^\beta(t)|e^{\delta_0 t}z_t|^2 \leq \frac{\nu}{4}\tau^\beta(t)\|e^{\delta_0 t}z_t(t)\|^2, \text{ for } \beta \geq 0, t \geq 0.$$

Hence, (6.6) yields that for all $\beta \geq 0$ and $t > 0$,

$$\frac{d}{dt}(\tau^\beta(t)|e^{\delta_0 t}z(t)|^2) \leq \delta\tau^\beta(t)e^{2\delta_0 t}J(t;z,z_t(t)) + \frac{4}{\nu}\tau^\beta(t)\|e^{\delta_0 t}F_t(t)\|_{-1}^2$$
$$+ \frac{4}{\nu}c_0^2\tau^\beta(t)\|z\|^2|e^{\delta_0 t}z_t|^2\,. \tag{6.7}$$

Integration of (6.7) for t from 0 to t leads to that for all $t \geq 0$

$$\tau^\beta(t)|e^{\delta_0 t}z_t(t)|^2 \leq \tau^\beta(0)|z_{t0}|^2 + 2\delta\int_0^t \tau^\beta(s)e^{2\delta_0 s}J(s;z,z_t(s))ds$$
$$+ \frac{4}{\nu}\int_0^t \tau^\beta(s)\|e^{\delta_0 t}F_t(s)\|_{-1}^2 ds + \frac{4}{\nu}c_0^2\int_0^t \tau^\beta(s)\|z\|^2|e^{\delta_0 s}z_t(s)|^2 ds. \tag{6.8}$$

By using (4.8), Lemma 3.2-Lemma 3.3, we can derive from (6.8) that for all $t \geq 0$

$$\tau^\beta(t)|e^{\delta_0 t}z_t(t)|^2 \leq (1+\eta)\tau^\beta(0)|z_{t0}|^2 + \rho\delta\tau^\beta(t)\|e^{\delta_0 t}z(t)\|^2$$
$$+ \rho\delta\int_0^t \Big(2\alpha_0\tau^\beta(s) - \frac{d}{ds}\tau^\beta(s)\Big)e^{-2\alpha_0 s}\|\int_0^s e^{\delta\tau}z(\tau)d\tau\|^2 ds$$
$$+ \rho\delta\tau^\beta(t)e^{-2\alpha_0 t}\|\int_0^t e^{\delta\tau}z(\tau)\|^2 + 2\rho\delta\alpha_0\int_0^t \tau^\beta(s)\|e^{\delta_0 s}z(s)\|^2 ds$$
$$+ \frac{4}{\nu}\int_0^t \tau^\beta(s)\|e^{\delta_0 s}F_t(s)\|_{-1}^2 ds + \frac{4}{\nu}c_0^2\int_0^t \|z\|^2\tau^\beta(s)|e^{\delta_0 s}z_t(s)|^2 ds \tag{6.9}$$
$$\leq c\Big[\tau^\beta(0)(\|z_0\|^2 + |z_{t0}|^2) + G_\beta(t) + \frac{4}{\nu}\int_0^t \tau^\beta(s)\|e^{\delta_0 s}F_t(s)|^2 ds\Big]$$
$$+ \frac{4}{\nu}c_0^2\int_0^t \tau^\beta(s)\|z\|^2|e^{\delta_0 s}z_t(s)|^2 ds.$$

From (6.9), (5.7), (2.14) and Theorem 1.3 it follows that for all $\beta \geq 0, t \geq 0$

$$\tau^\beta(t)|e^{\delta_0 t}z_t(t)|^2 \leq c\Big[\tau^\beta(0)(\|z_0\|^2 + |z_{t0}|^2) + G_\beta(t)$$

$$+\frac{4}{\nu}\int_0^t \tau^\beta(s)\|e^{\delta_0 s}F_t(s)\|_{-1}^2 ds + e^{2\alpha_1 t}\Big] \le c\Big[\|z_0\|^2 + |z_{t0}|^2 + e^{2\alpha_1 t}\Big]. \quad (6.10)$$

This completes the proof of Theorem 6.1. □

Theorem 6.2. *The solution* (z, r) *of the system* (2.8) *satisfies the following estimates:*

$$\tau^\beta(t)|e^{\delta_1 t}Az(t)|^2 \le c, \quad \forall t \ge 0, \ \beta \ge 0, \quad (6.11)$$

for some $c_8 > 0$.

Proof. From (2.8) it follows that

$$\begin{aligned}(e^{\delta_0 t}z_t, e^{\delta_0 t}Az) &+ \epsilon|e^{\delta_0 t}Az|^2 + J(t; z, e^{2\delta_0 t}Az(t)) + b(z, e^{\delta_0 t}z, e^{\delta_0 t}Az)\\ &+ b(\bar{u}, e^{\delta_0 t}z, e^{\delta_0 t}Az) + b(e^{\delta_0 t}z, \bar{u}, e^{\delta_0 t}Az) \quad (6.12)\\ &= (e^{\delta_0 t}F, e^{\delta_0 t}Az) \le \frac{\epsilon}{4}|e^{\delta_0 t}Az|^2 + \epsilon^{-1}|e^{\delta_0 t}F(t)|^2 .\end{aligned}$$

From (2.4) one can derive that

$$\begin{aligned}&|b(\bar{u}, e^{\delta_0 t}z, e^{\delta_0 t}Az)| + |b(e^{\delta_0 t}z, \bar{u}, e^{\delta_0 t}Az)|\\ &\quad \le c_0\lambda_1^{-1/4}\|\bar{u}\|^{1/2}|A\bar{u}|^{1/2}\|e^{\delta_0 t}z\||e^{\delta_0 t}Az| \le \frac{\epsilon}{8}|e^{\delta_0 t}Az|^2 + \frac{\nu}{4}\lambda_1\eta\|e^{\delta_0 t}z\|^2,\\ &|b(z, e^{\delta_0 t}z, e^{\delta_0 t}z)| \le c_0|z|^{1/2}\|z\|^{1/2}\|e^{\delta_0 t}z\|^{1/2}|e^{\delta_0 t}Az|^{3/2}\\ &\quad \le \frac{\epsilon}{8}|e^{\delta_0 t}Az|^2 + (\frac{2}{\epsilon})^3 c_0^4|z|^2\|z\|^2\|e^{\delta_0 t}z\|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}2(z_t, Az) &+ \epsilon|e^{\delta_0 t}Az(t)|^2 + 2J(t; z, e^{2\delta_0 t}Az(t))\\ &\le \frac{\nu}{2}\eta\lambda_1\|e^{\delta_0 t}z\|^2 + \frac{2}{\epsilon}|e^{\delta_0 t}F|^2 + g(t)\|e^{\delta_0 t}\|z\|^2, \ \forall t \ge 0. \quad (6.13)\end{aligned}$$

Note

$$2|(z_t, Az)| \le \frac{\epsilon}{4}|Az|^2 + \frac{1}{\epsilon}|z_t|^2,$$

$$2|J(t; z, e^{2\delta_0 t}Az(t))| \le \frac{\epsilon}{4}|e^{\delta_0 t}Az|^2 + \frac{\rho^2}{\epsilon}e^{-2\alpha_0 t}|\int_0^t e^{\delta\tau}Az(\tau)d\tau|^2.$$

For all $t \ge 0$, $\beta \ge 0$,

$$\begin{aligned}\frac{\epsilon}{2}\tau^\beta(t)|e^{\delta_0 t}Az(t)|^2 &\le \frac{8}{\epsilon}\tau^\beta(t)|e^{\delta_0 t}z_t(t)|^2 + \frac{8}{\epsilon}\rho^2\tau^\beta(t)e^{-2\alpha_0 t}|\int_0^t e^{\delta\tau}Az(\tau)d\tau|^2\\ &+ \frac{2}{\epsilon}\tau^\beta(t)|e^{\delta_0 t}F(t)|^2 + g(t)\tau^\beta(t)\|e^{\delta_0 t}z\|^2, \ \forall t \ge 0.\end{aligned}$$

Or, by using Lemma 3.3, one can obtain another estimate

$$
\begin{aligned}
&\frac{\epsilon}{2}\tau^{\beta}(t)|e^{\delta_0 t}Az(t)|^2 \\
&\le \frac{8}{\epsilon}\tau^{\beta}(t)|e^{\delta_0 t}z_t(t)|^2 + \frac{2}{\epsilon}\tau^{\beta}(t)|e^{\delta_0 t}F(t)|^2 + g(t)\tau^{\beta}(t)\|e^{\delta_0 t}z\|^2 \\
&+ c\Big[\tau^{\beta}(0)\|z_0\|^2 + G_{\beta}(t)\Big] + \frac{8\rho}{\epsilon}\int_0^t \tau^{\beta}(s)g(s)\|e^{\delta_0 s}z(s)\|^2 ds \\
&\le c\Big[\tau^{\beta}(t)|e^{\delta_0 t}z_t|^2 + \|z_0\|^2 + \tau^{\beta}(t)|e^{\delta_0 t}F(t)|^2 \\
&+ g(t)\tau^{\beta}(t)\|e^{\delta_0 t}z(t)\|^2 + G_{\beta}(t) + \int_0^t g(s)\tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2 ds\Big],
\end{aligned}
\tag{6.14}
$$

for all $t \ge 0$. From Lemma 3.2, (5.5)-(5.7) and (2.13) it follows that

$$
\begin{aligned}
&\tau^{\beta}(t)|e^{\delta_0 t}F(t)|^2 + g(t)\tau^{\beta}(t)\|e^{\delta_0 t}z(t)\|^2 + G_{\beta}(t) \\
&\qquad + \int_0^t g(s)\tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2 ds \le ce^{2\alpha_1 t}, \forall t \ge 0, \beta \ge 0.
\end{aligned}
\tag{6.15}
$$

Combining (6.14)-(6.15) with (6.1) yields (6.11). □

Theorem 6.3. *Under the assumptions of Theorem* 1.4, *the solution* (z, r) *of problem* (2.8) *satisfies that for all* $t \ge 0$

$$
\tau^{\beta}(t)\|e^{\delta_1 t}r(t)\|^2_{H^1(\Omega)} \le c_9,\ \forall \beta \ge 0,\ t \ge 0 \tag{6.16}
$$

for some $c_9 > 0$.

Proof. From (2.8), Lemma 3.3 it follows that

$$
\begin{aligned}
&\tau^{\beta}(t)|e^{\delta_0 t}\nabla r(t)|^2 \le c\tau^{\beta}(t)\Big[|e^{\delta_0 t}z_t(t)|^2 + |e^{\delta_0 t}Az(t)|^2 + g(t)\|e^{\delta_0 t}z\|^2 \\
&+ e^{-2\alpha_0 t}|\int_0^t e^{\delta\tau}Az(\tau)d\tau|^2 + |e^{\delta_0 t}F(t)|^2\Big] \\
&\le c\tau^{\beta}(t)\big(|e^{\delta_0 t}z_t|^2 + |e^{\delta_0 t}Az(t)|^2\big) + c\tau^{\beta}(t)|e^{\delta_0 t}F(t)|^2 + c\tau^{\beta}(t)g(t)\|e^{\delta_0 t}z(t)\|^2 \\
&+ c\Big(\|z_0\|^2 + \int_0^t g(s)\tau^{\beta}(s)\|e^{\delta_0 s}z(s)\|^2 ds\Big).
\end{aligned}
\tag{6.17}
$$

Moreover, by using (2.1) and (3.2), we obtain

$$
\gamma\tau^{\alpha}(t)|e^{\delta_0 t}r(t)| \le \sup_{v\in X}\frac{d(v, \tau^{\alpha}(t)e^{\delta_0 t}r(t))}{\|v\|} \le \lambda_1^{-1/2}\tau^{\alpha}(t)|e^{\delta_0 t}\nabla r(t)|,
$$

which and (6.17) have given

$$\tau^\beta(t)\|e^{\delta_1 t}r(t)\|^2_{H^1(\Omega)} \le c\tau^\beta(t)\big(|e^{\delta_1 t}z_t|^2 + |e^{\delta_1 t}Az(t)|^2\big)$$
$$+ c\tau^\beta(t)|e^{\delta_1 t}F(t)|^2 + c\tau^\beta(t)g(t)\|e^{\delta_1 t}z(t)\|^2 + ce^{-2\alpha_1 t}\|z_0\|^2 + cG_\beta(t) \quad (6.18)$$
$$+ ce^{-2\alpha_1 t}\int_0^t g(s)\tau^\beta(s)\|e^{\delta_0 s}z(s)\|^2 ds.$$

From Theorem 6.1, Theorem 6.2, (6.18) and (2.13) it follows that

$$\tau^\beta(t)\|e^{\delta_1 t}r(t)\|^2_{H^1(\Omega)} \le c\Big[1 + G_\beta(t) + \tau^\beta(t)g(t)\|e^{\delta_1 t}z(t)\|^2 + e^{-2\alpha_1 t}\int_0^t g(s)\tau^\beta(s)\|e^{\delta_0 s}z(s)\|^2 ds\Big], \quad (6.19)$$

which and (6.15) yield (6.16). □

Finally, Theorem 1.4 is the consequence of Theorems 6.1–6.3.

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