Physical universals in problem of precursor soliton generation*

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Abstract Six physical universals and two general relations in the problem of locally forced precursor soliton generation are found theoretically in terms of the AfKdV equation derived by authors. These six universals and two general relations are examined by experiment and numerical calculation of two-layer flow based on the canonical character of the coefficients of the fKdV equations. From comparisons among the theoretical, numerical and experimental results, it is shown that they are in good agreement. There is not any free parameter in this theory, so the theory of the present paper can be used to predict the wave properties of locally forced precursor soliton generation.

Keywords: precursor soliton, fKdV equation, AfKdV equation, physical universal.

In the last decade, the problem of the precursor soliton generation (a sort of nonlinear generated waves by forced sources) has been the subject of many theoretical, experimental and numerical studies due to its real significance in meteorology, oceanography, hydrodynamics and other disciplines. Many authors have devoted themselves to the study of the precursor soliton generation along the frontier of this field to obtain a predictable theory. However there exist two free parameters in the previous theories; the intensity of the forced source and the level in the depressed region h_1. It is well known that the generating wave properties of the precursor solitons depend on the size and velocity of the forced sources and depend on the stratification if the fluid is stratified. For the same forced source and different velocities there exist a large number of the levels in the depressed region h_1; similarly for the same velocity and different forced sources there exist also a large number of the levels in the depressed region h_1. Unfortunately some of the previous theories are just dependent on this unknown parameter h_1, at the same time the intensity of the forced sources was not given in the previous theories, therefore those theories cannot predict the generating wave properties of the precursor solitons.

The present work is a further study on the theory of the precursor soliton generation. Our goal in this study is twofold. First of all, we attempt to construct a theory of the precursor soliton generation without any free parameter. Second, we attempt to discover if there exist invariant constants, i.e. physical universals, in the problem of the precursor soliton generation.

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1 Derivation of averaged fKdV equation

In this section, a transform among the frame of Shen[1] (x, t), the body frame and the absolute frame of Lee *et al.*[2] is discussed first, the phase frame in the absolute frame is then introduced to derive an averaged fKdV equation.

Shen[1] derived the fKdV equation of two-layer flow over the compact forced source as

$$m_1 \eta^{(1)}_{,t} + m_2 \eta^{(1)}_{,x} + m_3 \eta^{(1)} \eta^{(1)}_{,x} + m_4 \eta^{(1)}_{,xxx} = f(x)_{,x}/2, \tag{1}$$

where m_1, m_2, m_3, m_4 and $f(x)$ are the coefficients of the equation and the forced source respectively, which is expressed by a given sum of the topography $h(x)$ and the forced pressure $\bar{P}(x)$, $\eta^{(1)}$ is first-order approximation of the surface wave elevation η, and $\eta = \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + O(\varepsilon^3)$. Based on the canonical character of the coefficients of the fKdV equations, (1) is also a general form of the fKdV equation. In order to transform eq. (1) in the Gardner and Morikawa's frame (GM) into the body frame and absolute frame, we can introduce a laboratory dimensionless frame x', y', t' (LDC), topography h' and surface pressure $\bar{P}'$, namely,

$$x' = x^*/H, y' = y^*/H, t' = t^*/\sqrt{H/g}, \ h' = h^*(x^*)/H, \bar{P}' = \bar{P}^*/\rho g H, \tag{2}$$

we thus have the following relations between GM and LDC frames:

$$\partial/\partial x' = \varepsilon^{1/2}\partial/\partial x, \ \partial/\partial t' = \varepsilon^{3/2}\partial/\partial t. \tag{3}$$

Substituting (3) into (1) yields

$$m_1 \eta^{(1)}_{s,t'} + \varepsilon m_2 \eta^{(1)}_{s,x'} + m_3 \eta^{(1)}_s \eta^{(1)}_{s,x'} + m_4 \eta^{(1)}_{s,x'x'x'} = \varepsilon^2 f(x')_{,x'}/2. \tag{4}$$

For near-resonant flow $C = C_0 + \varepsilon\lambda$, in which C is the velocity of the flow, C_0 the critical velocity, λ the detoning parameter and ε the small parameter. For the problem of the precursor soliton generation, an equality $m_2 = m_1\lambda = m_1(C - C_0)/\varepsilon$ can also be proved to hold. Using the small forced source assumption $f(x) = \varepsilon^{-2} f(x')$ (see ref. [1]) and above conditions, we obtained the fKdV equation in the body frame as

$$m_1 \eta_{,t'} + m_1(C - C_0)\eta_{,x'} + m_3 \eta\eta_{,x'} + m_4 \eta_{,x'x'x'} = f(x')_{,x'}/2. \tag{5}$$

Introducing Galilean transform $x' = x'' + Ct''$, $t' = t''$, where x'', t'' are the absolute frame, (5) becomes the fKdV equation in the absolute frame as

$$m_1 \eta_{,t''} - m_1 C_0 \eta_{,x''} + m_3 \eta\eta_{,x''} + m_4 \eta_{,x''x''x''} = f(x'' + Ct'')_{,x''}/2, \tag{6}$$

where C also denotes the moving velocity of the forced source in the body and absolute frames. In the following the superscripts in (5) and (6) will be omitted. It should be also pointed out that there is not any small parameter in (5) and (6), which appears in the fKdV equation of Lee *et al.*[2].

Based on the theory of Xu *et al.*[3], we know that after the phase frame $\theta = x + Ct$ is introduced into (6), the wave properties of the precursor soliton generation can be found in this frame. Following the mean procedure of Xu *et al.*[3] $\overline{(\ \ )} = \frac{1}{\theta_2 - \theta_1}\int_{\theta_1}^{\theta_2}(\ \ )\mathrm{d}\theta$, the regional mean fKdV equation upstream is

$$M_2\bar{\eta} + m_3\bar{\eta}^2/2 + m_4\bar{\eta}_{,\theta\theta} = \bar{f}(\theta)/2 + \bar{C}, \ \theta \in (\theta_{\mathrm{up}}, \bar{\theta}), \tag{7}$$

where $\bar{\eta}$ and $\bar{f}(\theta)$ are the regional mean value of η and $f(\theta)$ in the upstream region, $M_2 = m_1(C - C_0)$ and $\bar{C}$ are the relative Froude number and the integral constant respectively. And the regional mean equation downstream is

$$M_2\bar{\eta} + m_3\bar{\eta}^2/2 + m_4\bar{\eta}_{,\theta\theta} = \bar{C}, \quad \theta \in (\bar{\theta}, \theta_{\rm down}), \tag{8}$$

where $\theta_{\rm up}$, $\theta_{\rm down}$ and $\bar{\theta}$ are the front edge point in the upstream of the forced source, the zero-crossing point of the trailing wavetrain behind the forced source and the crossing point between the wave and static level in vicinity of the forced source respectively. Under the asymptotic sense, for a new generating precursor soliton the forced source $\bar{f}(\theta)$ basically remains, so it can be approximated as $\bar{f}(\theta) = 2p\delta(\theta)$. Hence when $t \to \infty$, we can obtain an averaged fKdV equation as follows:

$$M_2\bar{\eta}_{,\theta} + m_3\overline{\eta\eta_{,\theta}} + m_4\bar{\eta}_{,\theta\theta\theta} = 0, \; (-\infty < \theta < \infty), \tag{9}$$

$$\bar{\eta}(\theta \to \pm\infty) = H_\pm, \bar{\eta}_{,\theta}(\pm\infty) = \bar{\eta}_{,\theta\theta}(\pm\infty) = 0,$$
$$\bar{\eta}_{,\theta}(0_-) = \bar{\eta}_{,\theta}(0_+) = p/m_4, \tag{10}$$

where 0_- and 0_+ are the left and right zeros of $\delta(0)$ respectively, H_- and H_+ are defined as

$$H_- = \frac{1}{0_- - \theta_{\rm up}}\int_{\theta_{\rm up}}^{0_-} \eta {\rm d}\theta, \; H_- = \frac{1}{\theta_{\rm down} - 0_+}\int_{0_+}^{\theta_{\rm down}} \eta {\rm d}\theta. \tag{11}$$

2 Asymptotic mean wave elevation and pseudo mean wave resistance

To determine the asymptotic mean characters of the fKdV equation, asymptotic mean hydraulic falls at the subcritical cutoff points must be found first. Integrating (9) and using (10) yields

$$M_2\bar{\eta} + m_3\bar{\eta}^2/2 + m_4\bar{\eta}_{,\theta\theta} = M_2H_- + m_3H_-^2/2. \tag{12}$$

Introducing a transform

$$\bar{\eta} = \xi + H_-, \tag{13}$$

(12) becomes an ordinary differential equation on ξ with boundary conditions as

$$\mathscr{P}\xi + m_3\xi^2/2 + m_4\xi_{,\theta\theta} = 0, \; \xi(0_-) = 0, \xi_{,\theta}(-\infty) = 0, \; \xi_{,\theta}(0_-) = p/m_4, \tag{14}$$

where

$$\mathscr{P} = M_2 + m_3H_-. \tag{15}$$

Multiplying the first equation of (14) with $\xi_{,\theta}$ and integrating the result and employing (15) yields

$$p^2/m_4 - \mathscr{P}\xi^2 - \frac{m_3}{3}\xi^3 = 0. \tag{16}$$

By a transform

$$\xi = Z - \mathscr{P}^2/m_3, \tag{17}$$

it follows that

$$Z^3 - 3Z(\mathscr{P}m_3^{-1})^2 + Y = 0, \tag{18}$$

where

$$Y = 2(\mathscr{P}m_3^{-1})^3 - 3m_3^{-1}m_4^{\,1}p^2. \tag{19}$$

For subcritical state ($M_2 < 0$), there is at least a critical value of $\mathscr{P}$ in (18), i.e. $\mathscr{P}_L$, so that eq. (18) has a solution of asymptotic mean hydraulic falls. The subcritical value of $\mathscr{P}$ is obtained as[1,3]

$$\mathscr{P}_L = (3m_3^2p^2/4m_4)^{1/3} = M_{2L} + m_3\overset{\circ}{H}, \tag{20}$$

and

$$\overset{\circ}{H}_- = \mathscr{F} - M_{2L}/m_3, \; \mathscr{F} = \mathscr{P}_L/m_3 = (3p^2/4m_3m_4)^{1/3}, \tag{21}$$

where $\mathscr{F}$ is the half-hydraulic fall, M_{2L} is the value of M_2 at the subcritical cutoff point C_L, $\mathring{H}$ is the upstream asymptotic mean level at the subcritical cutoff point. A solution of (9) is

$$\bar{\eta} = \mathring{H}_- + 2\mathscr{F}\{-1 + 3\operatorname{sech}^2[(M_{2L}/4m_4)^{1/2}(\theta - \theta_0)]/2\}, \tag{22}$$

where

$$\theta_0 = 2(m_4/M_{2L})^{1/2}\operatorname{sech}^{-1}(2/3)^{1/2}. \tag{23}$$

Let $\mathring{H}_+$ be the asymptotic mean level of the depressed region at the subcritical cutoff point, and generally $\mathring{H}_+ < 0$. Thus when $\theta \to +\infty$, (22) has a limit,

$$\mathring{H}_- - \mathring{H}_+ = 2\mathscr{F}, \quad \mathring{H}_+ = -\mathscr{F} - M_{2L}/m_3. \tag{24}$$

In the near resonant region, the asymptotic mean elevations upstream and downstream can be determined. Substituting (10) into (7) yields

$$M_2 = -m_3(H_- + H_+)/2, \quad \bar{C} = -m_3(H_- H_+)/2. \tag{25}$$

Xu *et al*.[3] proved that the solutions of the asymptotic mean elevations upstream and downstream are unique in the AfKdV system and have the following forms:

$$H_- = \mathscr{F} - m_1(C - C_0)/m_3, \quad H_+ = -\mathscr{F} - m_1(C - C_0)/m_3. \tag{26}$$

Multiplying (7) with $\bar{\eta}_{,\theta}$ and integrating the result from $-\infty$ to ∞ yields

$$\bar{D} = -\frac{1}{2}\int_{-\infty}^{\infty}\bar{f}(\theta)\bar{\eta}_{,\theta}\mathrm{d}\theta = -m_3(H_- - H_+)^3/12 = -2m_3\mathscr{F}^3/3 = -p^2/2m_4, \tag{27}$$

where $\bar{D}$ is defined as the pseudo mean wave resistance and can also be expressed in terms of (26), namely,

$$\begin{aligned} \bar{D}_- &= -2m_3(H_- + m_1(C - C_0)m_3^{-1})^3/3, \\ \bar{D}_+ &= 2m_3(H_+ + m_1(C - C_0)m_3^{-1})^3/3, \end{aligned} \tag{28}$$

where $\bar{D}_-$ and $\bar{D}_+$ are the pseudo mean wave resistances expressed by the asymptotic mean elevations upstream and downstream respectively. Generally $\bar{D} = \bar{D}_+ = \bar{D}_-$, particularly when $C = C_0$, the relation $\bar{D} = \bar{D}_+ = \bar{D}_-$ also holds.

3 Theoretical mean wave resistance and regional energy distribution

In this section the conservation of mass and energy is discussed, and the regional energy distribution is also given by using the solution of the asymptotic mean hydraulic fall at the subcritical cutoff point. From (5) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\eta\mathrm{d}x = 0, \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\frac{1}{2}\eta^2\mathrm{d}x = -\int_{-\infty}^{\infty}\frac{1}{2m_1}f(x)\eta_{,x}\mathrm{d}x, \tag{30}$$

(29) and (30) are the mass and energy conservation theorems respectively as defined by Xu *et al*.[3]. Let the total mass of the system be

$$M = \int_{-\infty}^{\infty}\eta\mathrm{d}x. \tag{31}$$

Based on the symmetry of the wave shape of the trailing wavetrain, it is also known that

$$\mathring{M}_{\text{up}} = \mathring{M}_{\text{down}}, \tag{32}$$

where $\mathring{M}_{\text{up}}$ and $\mathring{M}_{\text{down}}$ are the time rate of change of mass in the upstream and depressed region respectively, $(\overset{\circ}{\ })$ stands for the time derivative. They can be expressed asymptotically as

$$\overset{\circ}{M}_{\mathrm{up}} = -M_2 m_1^{-1} H_- - m_3 m_1^{-1} H_-^2/2, \quad \overset{\circ}{M}_{\mathrm{down}} = -M_2 m_1^{-1} H_+ - m_3 m_1^{-1} H_+^2/2. \tag{33}$$

Define the wave resistance as

$$D = -\int_{-\infty}^{\infty} \frac{1}{2m_1} f(x)\eta_{,x}\mathrm{d}x, \tag{34}$$

generally $D \neq \bar{D}$. Its mean value can be defined as the theoretical mean wave resistance[3], namely,

$$\langle D\rangle = \bar{D}/m_1 = -2m_3\mathscr{F}^3/3m_1. \tag{35}$$

From (30), the mean energy conservation is obtained as

$$\overset{\circ}{E} = \overset{\circ}{E}_1 + \overset{\circ}{E}_2 + \overset{\circ}{E}_3 = \langle D\rangle, \tag{36}$$

in which $\overset{\circ}{E}_1$, $\overset{\circ}{E}_2$, $\overset{\circ}{E}_3$ and $\overset{\circ}{E}$ are the time mean rate of the energy in the generating region of the precursor soliton, in the depressed region, in the generating region of the trailing wavetrain and in the whole region respectively. They can be expressed as

$$\overset{\circ}{E}_1 = -M_2 H_-^2/(2m_1) - m_3 H_-^3/(3m_1), \tag{37}$$

$$\overset{\circ}{E}_2 = -M_2 H_+^2/(2m_1) - m_3 H_+^3/(3m_1). \tag{38}$$

By the definition of M_2, the regional mean energy can also be expressed as

$$\overset{\circ}{E}_1 = -m_3(\mathscr{F} - M_2 m_3^{-1})^2(2\mathscr{F} + M_2 m_3^{-1})/6m_1, \tag{39}$$

$$\overset{\circ}{E}_2 = m_3(\mathscr{F} + M_2 m_3^{-1})^2(2\mathscr{F} - M_2 m_3^{-1})/6m_1, \tag{40}$$

$$\begin{aligned}\overset{\circ}{E}_3 = &-2m_3\mathscr{F}^3/3m_1 + m_3(\mathscr{F} - M_2 m_3^{-1})^2(2\mathscr{F} + M_2 m_3^{-1})/6m_1 \\ &- m_3(\mathscr{F} + M_2 m_3^{-1})^2(2\mathscr{F} - M_2 m_3^{-1})/6m_1.\end{aligned} \tag{41}$$

From (35), (39)—(41), we know that when the moving velocity of the forced source is at the resonant point, i.e. $C = C_0$ or $M_2 = 0$, four universals, which do not depend on the intensity of the forced source and the stratification, of the energy ratios

$$\overset{\circ}{E}_1/\overset{\circ}{E} : \overset{\circ}{E}_2/\overset{\circ}{E} : \overset{\circ}{E}_3/\overset{\circ}{E} : \langle D\rangle/\overset{\circ}{E} = (1/2) : (-1/2) : 1 : 1 \tag{42}$$

are obtained in r.h.s. of eq. (42), they will be examined in terms of the numerical calculation.

4 Two general relations and universals of velocity and width

By the previous theory, the developed precursor soliton can be considered as a free KdV soliton. However for the free KdV solitons, the mass and energy conservation theorems must be satisfied asymptotically. To find the wave properties, we use initial conditions $\eta(\pm\infty, 0) = 0$, $\eta_{,x}(\pm\infty, 0) = 0$ and eq. (5) without the source term. Introducing a replacement

$$\xi = \eta + M_2/m_3 \tag{43}$$

and transforming eq. (5) and the initial conditions yields

$$\begin{aligned}&\xi_{,t} + m_3 m_1^{-1}\xi\xi_{,x} + m_4 m_1^{-1}\xi_{,xxx} = 0, \\ &\xi(\pm\infty) = M_2 m_3^{-1}, \quad \xi_{,x}(\pm\infty) = 0,\end{aligned} \tag{44}$$

a solution of (44) is

$$\eta = \xi - M_2/m_3 = -3(M_2 - m_1 U_-)m_3^{-1}\mathrm{sech}^2((M_2 - U_- m_1)^{1/2}$$

$$\times(-4m_4)^{-1/2}(x + x_0 - U_- t)). \tag{45}$$

in which x_0 is the initial phase of η, U_- is the moving velocity of the precursor solitons. We can suppose that the total energy of the precursor solitons upstream equals the sum of the energy of single free KdV solitons. Assume the mass of single soliton is $\overset{*}{m}$, from (45) we have

$$\overset{*}{m} = -6(M_2 - m_1 U_-)^{1/2} m_3^{-1}(-4m_4)^{1/2}. \tag{46}$$

Let τ be the generating period of the single soliton, we then obtain

$$\overset{*}{m} = \tau(-M_2 m_1^{-1} H_- - m_3 m_1^{-1} H_-^2/2). \tag{47}$$

Let $\overset{*}{\varepsilon}$ be the energy of the single soliton, by (45) we have

$$\overset{*}{\varepsilon} = -6(M_2 - m_1 U_-)^{3/2} m_3^{-2}(-4m_4)^{1/2}, \tag{48}$$

and

$$\overset{*}{\varepsilon} = \tau(-M_2 H_-^2/(2m_1) - m_3 H_-^3/(3m_1)). \tag{49}$$

Assume that the generated amplitude and the width of the single soliton are $\mathscr{A}$ and $\mathscr{L}$ respectively, we then obtain

$$\mathscr{A} = -3(M_2 - m_1 U_-) m_3^{-1}, \quad \mathscr{L} = (-(M_2 - m_1 U_-)/4m_4)^{-1/2}, \tag{50}$$

and $\overset{*}{m}$ and $\overset{*}{\varepsilon}$ can be expressed in terms of $\mathscr{A}$ and $\mathscr{L}$ as

$$\overset{*}{m} = 2\mathscr{A}\mathscr{L}^{-1}, \quad \overset{*}{\varepsilon} = 2\mathscr{A}^2(3\mathscr{L})^{-1}. \tag{51}$$

From the above results, we can find that the generated amplitude and period of the precursor solitons are respectively

$$\begin{aligned} \mathscr{A} &= 2(\mathscr{F} + m_1(C - C_0)m_3^{-1}/2)(\mathscr{F} - m_1(C - C_0)m_3^{-1})(m_3\mathscr{F} + m_1(C - C_0))^{-1}, \\ \tau &= -8(3\mathscr{A}m_3^{-1}m_4)^{1/2}(C - C_0 + m_3 m_1^{-1}\mathscr{F})^{-1}(\mathscr{F} - m_1(C - C_0)m_3^{-1})^{-1}. \end{aligned} \tag{52}$$

From (52), we can obtain two general relations of the generating amplitude and period, i. e. when $C = C_0$, $\mathscr{A} = 2\mathscr{F}$ and $\tau = -8m_1 m_3^{-1}\sqrt{6m_4 m_3^{-1}}\mathscr{F}^{-3/2}$. These two general relations will be examined in the next sections numerically and experimentally. Another expression of the width of the single soliton is

$$\mathscr{L} = 2\sqrt{3}(m_4\mathscr{A}^{-1}m_3^{-1})^{1/2}, \tag{53}$$

from (50) and (53), we can find the moving velocity of the precursor soliton upstream U_- is

$$\begin{aligned} U_- &= 2m_3((\mathscr{F} + m_1(C - C_0)/2m_3)^2/3m_1 + 3(m_1(C - C_0)/m_3)^2/4) \\ &\quad \times(\mathscr{F} + m_1(C - C_0)/m_3)^{-1}. \end{aligned} \tag{54}$$

The velocity of the first zero crossing of the trailing wavetrain is

$$U_+ = -m_3(\mathscr{F} - m_1(C - C_0)/m_3)/2m_1. \tag{55}$$

From (54) and (55), it was known that when the velocity of the forced source is at the resonant points, i.e. $C = C_0$, we can obtain

$$\widetilde{U}_-/\widetilde{U}_+ = \mathscr{D}_-/\mathscr{D}_+ = -4/3, \tag{56}$$

where $\widetilde{U}_-$ and $\widetilde{U}_+$ are the velocities of the precursor solitons and of the first zero crossing of the trailing wavetrain at the resonant points. $\mathscr{D}_-$ and $\mathscr{D}_+$ are the widths of the generating region of the precursor solitons upstream and of the depressed region downstream at the resonant points. The ratio $-4/3$ is called the universals of the velocity and width in this paper due to the independence with the intensity of the forced source and the stratification. It was also known that the obtained results all depend on $\mathscr{F}$, and $\mathscr{F}$ depends on the intensity of the forced source p. To examine the above results with the experiment and numerical calculation of two-layer flow forced by the semicircular topography, the intensity of the semicircular topography must be determined. Following Xu *et al*.[4], it can be found theoretically as

$$p = \pi C_0^2 (C_0^2 (1+\sigma) - \sigma) \alpha^{3/2} / 4. \tag{57}$$

For non-semicircular topography, the intensity p can also be determined theoretically in terms of the area principle of Shen[1].

5 Numerical and experimental examination of universals

5.1 Numerical calculation of two-layer flow forced by the semicircular topography

To examine the two general relations and the universals, a numerical calculation of two-layer flow is carried out by using the fKdV eq. (6). The coefficients of (6) are given by Xu *et al*.[5]. The dimensionless semicircular topography is $h(x) = \sqrt{\alpha^2 - x^2}$, in which α is the dimensionless radius of the semicircular topography, the forced source is

$$f(x + Ct) = (C_0^2(1+\sigma) - \sigma) C_0^2 h(x + Ct). \tag{58}$$

The boundary and initial conditions are $\eta(\pm\infty) = 0$, $\eta_{,x}(\pm\infty) = \eta_{,xx}(\pm\infty) = 0$ and $\eta(x, 0) = 0$. The FDM is employed to calculate (6) numerically. Following the numerical method of Xu *et al*.[6], the obtained numerical results are moved in the direction of the moving topography for each time step. To simulate the generating phenomena of the precursor soliton the predictor-corrector difference equations and conservative numerical scheme are all adopted to reduce the numerical dissipation and dispersion and to ensure the conservation of total water mass. For a real numerical example of two-layer flow, the stratified parameters ρ, σ, γ must be predescribed and be in agreement with that used in the experiments. In our numerical calculation, the parameters are $\alpha = 0.2308$, $\rho = 0.835$, $\sigma = 1.03125$ and $\gamma = 1.0$, in which ρ is a ratio of the density in the upper layer to that in the lower layer, σ is a ratio of the depth of the upper layer to that of the lower layer, γ is a ratio of the velocity in the upper layer to that in the lower layer. By the given parameters, the critical velocity $C_0 = C_s^{(0)} = C_i^{(0)} = 0.9782$ can be calculated. By the given parameters, the numerical amplitude and period of the precursor solitons at the resonant point are 0.582 and 26.52 respectively, and the theoretical amplitude and period are 0.602 and 28.02 respectively, the theoretical and numerical mean wave resistance are 0.023 771 and 0.024 002 respectively. The numerical widths of the generating regions of the precursor solitons and of the depression are −42.5 and 31.5 at the resonant point respectively. From these results, it was shown that the theory and numerical calculation are in good agreement. The theoretical and numerical results are indicated in figs. 1—3. A comparison between the theoretical and numerical mean wave resistances and between the theoretical and numerical upstream energy is indicated in figure 1.

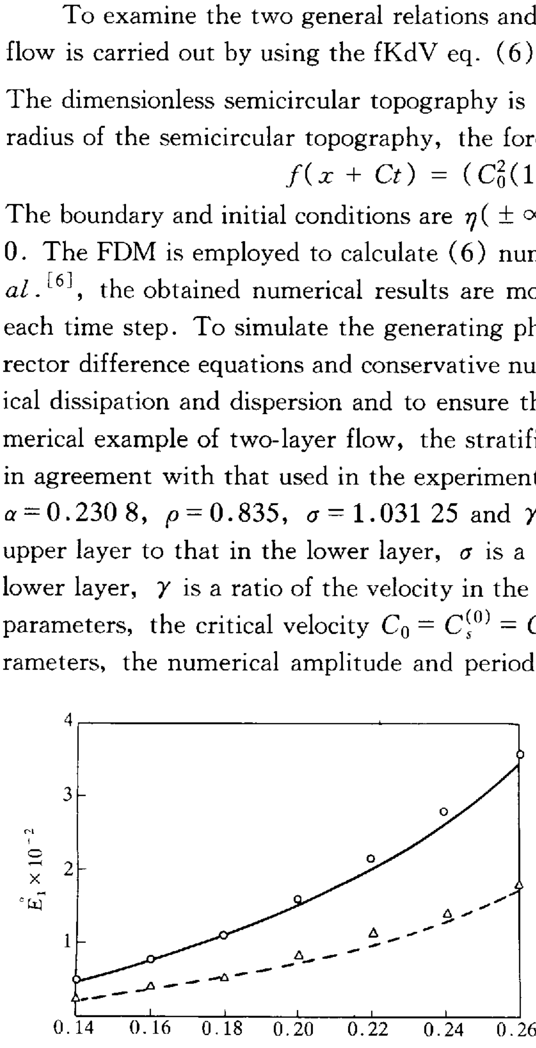

Fig. 1. An examination of the universals of the energy ratios. ——, Theoretical mean wave resistance $\langle D \rangle$; – – –, theoretical upstream energy $\overset{\circ}{E}_1$; △, numerical upstream energy; ○, numerical mean wave resistances. Parameters $\rho = 0.835$, $\sigma = 1.03125$, $\gamma = 1.0$ and α varies from 0.14 to 0.26.

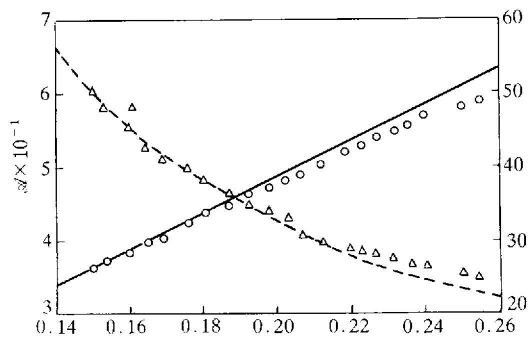

Fig. 2. An examination of two general relations on the amplitude and period. ——, Theoretical amplitudes; – – –, theoretical periods; ○, experimental amplitudes; △, experimental periods. Parameters $\rho = 0.835$, $\sigma = 1.031\ 25$, $\gamma = 1.0$ and α varies from 0.14 to 0.26.

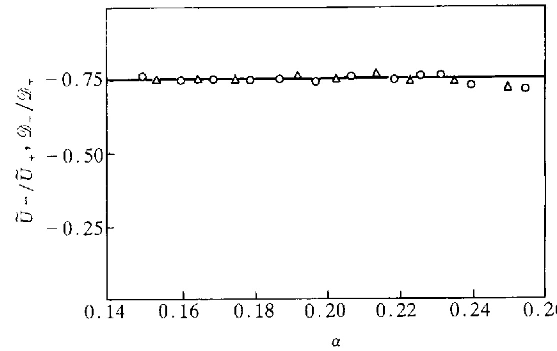

Fig. 3. An examination of the velocity and width universals. △, Experimental width ratios; ○, experimental velocity ratios; ——, theoretical results. Parameters $\rho = 0.835$, $\sigma = 1.031\ 25$, $\gamma = 1.0$ and α varies from 0.14 to 0.26.

5.2 Experiment of two-layer flow forced by the semicircular topography

An experiment of two-layer flow was carried out in Physical Oceanography Laboratory of Ocean University of Qingdao, China. The tank is 18 m long, 0.15 m wide and 0.33 m deep. The upper fluid is kerosine with density 0.829 g/cm^3, and the lower layer fluid is fresh water with density 0.993 g/cm^3. The tank system includes the body of the tank, the towing system of the models and the monitoring system of the velocity of the model. The varying range of the velocity of the model (or the topography) C is from 0.1 to 1.8. The instantaneous velocities of the model were measured by two optico-electric digit recorders (ODER) which were set in two given locations. The mean velocity is a mean value of the velocities at two locations. Generally the departure between the instantaneous velocities and the mean velocity is less than 0.5%. The wave pattern was recorded by two cameras RICOH-10 and NICON with fish eye lens. The RECOH-10 was used to record the developed second wave, and the NICON was used to record the developed third wave. From the experiments, it is known that the developed third wave is just the first precursor soliton. Assume RICOH-10 is at the location $L1$, NICON is at the location $L2$, the mean velocity of the model in the same run is C_M, the number of the precursor solitons is N, the generating period then can be calculated by $T^* = (L2 - L1)(C_M N)^{-1}$, the dimensionless generating period is $T = T^* H^{-1}\sqrt{gH}$. The generating amplitude of the precursor solitons can be read with the vertical rulers directly. An examination of the amplitude and period relations is indicated in figure 2.

The moving velocities of the precursor solitons U_- can be obtained with $U_- = -(U_1 - C_M)$, in which U_1 and C_M are the absolute velocities of the precursor solitons and the mean velocities of the model respectively. The mean velocities are calculated by $C_M = (C_1 + C_2)/2$, in which C_1 and C_2 are the velocities of the model at $L1$ and $L2$ respectively. The widths of the generating region of the precursor solitons and of the depression are $\mathcal{D}_- = (x_{up} - \bar{x})$ and $\mathcal{D}_+ = (x_{down} - \bar{x})$ respectively, which can be read from the photos directly. Fig. 3 is an examination on the universals of the velocity and width.

5.3 Conclusions

The theoretical results of this paper all can be reduced to that of Lee *et al*. (1989). For the single layer flow, there are $m_1=1$, $m_3=-3/2$, $m_4=-1/6$, $C_0=1$ and $C=F$, (5) becomes the fKdV equation of the single layer flow in the body frame to be defined by Lee *et al*.[2]. By our definitions $H_+=-\beta$, $M_2/m_3=-2(F-1)/3=-2\delta$, (24) thus becomes $\beta=\mathscr{F}-2\delta$, in which β and δ are the symbols used by Lee *et al*.[2]. Substituting this result into our theoretical formulae, we know that our results all can be reduced to that of single layer flow of Lee *et al*.[2]. It should be pointed out that (54) is not obtained by Lee *et al*.[2]. The universals of the velocity and the width can also be examined with the numerical calculation of the single layer flow, from fig. 2(b) of Lee *et al*.[2] it is known that $\widetilde{U}_-$ is equal to -0.185 and $\widetilde{U}_+$ is equal to 0.140 approximately, so $\widetilde{U}_-/\widetilde{U}_+=-1.321\,4\approx-4/3$. Obviously $\mathscr{D}_-/\mathscr{D}_+\approx-4/3$. The universals of the energy ratios can also be examined with the numerical results of the two-layer flow[4] directly. Finally it should be pointed out that due to the canonical character of the coefficients of the fKdV equations, the theoretical results of this paper hold for locally forced two-dimensional flow under the condition $\bar{f}(x)>0$, which is the most general case in the nature.

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