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BLOCKING OF SOLITARY PULSES IN A NONLINEAR FIBER

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By studying solutions to a forced cubic-nonlinear Schrödinger equation, we conclude that solitary pulse transmission in a fiber with an external disturbance (referred to as blocking) is possible only when the supplying power is sufficiently large. This result is justified theoretically by proving an existence and nonexistence theorem, and numerically by finding profiles of envelopes of the transverse electric field in the fiber.

1. Introduction

Pulse transmission in an optic fiber has attracted many people's attention in the last 25 years (see [1-4] and references therein). Due to the balance of dispersion and nonlinearity, the governing equation for the modulation of the electric waves is a cubic-nonlinear equation of Schrödinger type.

The independent variables of this equation are the longitudinal coordinate x and the time coordinate t. Recent development on the theoretical studies of optic fiber may be classified into the following two categories: (i) complex geometry of the fiber cross-section; and (ii) complex material structure of the fiber. In the category (i) (see [2]), the modulation of a pulse is governed by a cubic-nonlinear Schrödinger equation with constant coefficients. The solutions to this equation are translation invariant. That is, if $\varphi(x, t)$ is a solution, then $\varphi(x+a, t+b)$ is also a solution for any constants a and b. While in the category (ii), the governing equation is a cubic-nonlinear equation with variable coefficients. This equation is referred to as the cubic-nonlinear equation of Schrödinger type. Because the coefficients of the equation depend on x and t explicitly, the property of translation invariance is lost in the category (ii). The electric waves travel along a ray. This ray is no longer a straight line (see [1] and [6]).

The breaking of the translation invariance property may come about in many other ways. In this paper, we present a specific way, which is different from category (ii), to break translation invariance. This is to exert a moving disturbance to the fiber. The disturbance travels at the group velocity of the pulse being transmitted. This disturbance is called a blocking. The governing equation for the wave modulation is then a forced cubic-nonlinear Schrödinger equation (fCNLS). The forcing term on the right-hand side of the fCNLS is due to the blocking, which depends on x and t explicitly. For the solitary pulse to overcome the blocking, the intensity of the pulse must be sufficiently strong. In mathematical terms, if the intensity of the pulse is not strong enough, then that fCNLS has no solutions. It appears that such an intuitively obvious claim has not been rigorously justified yet. This paper is addressed to this problem.

2. Pulse blocking

The incoming solitary pulse is assumed to be an amplitude modulated plane wave $\exp[i(\kappa x-\omega t)]$, where x is the longitudinal coordinate of the fiber. The index of refraction of the fiber is

$$n=n_0(\omega)+n_2|\boldsymbol{E}|^2.$$

Here $\boldsymbol{E}$ is the electric field, and $|n_2/n_0| \ll 1$. The plane wave being amplitude modulated is assumed to be composed of harmonics near a monochromatic wave $\exp[i(k_0x - \omega_0 t)]$. Namely, $\omega = \omega_0 + \Omega$, $\kappa = k_0 + k$, and $|\Omega/\omega_0| \ll 1$, $|k/k_0| \ll 1$. The zeroth order asymptotic approximation of the dispersion relation for the monochromatic wave $\exp[i(\kappa_0 x - \omega_0 t)]$ is

$$\omega_0^2 n_0^2(\omega_0) = c^2 k_0^2$$

where c is the light speed. The exact dispersion relation $\omega = \omega(\kappa)$ is not known, but it is usually a smooth function [5]. We have the following approximation:

$$\omega = \omega(k_0 + k) = \omega(k_0) + \omega_0'(k_0)k + \tfrac{1}{2}\omega_0''(k_0)k^2 + \cdots = \omega_0 + \Omega.$$

Then the group velocity is

$$v_g = \frac{\partial\omega}{\partial k} \approx \omega_0'(k_0) + \omega_0''(k_0)k.$$

For a radially symmetric fiber, a transverse component of the electric field is assumed to be of the form

$$E(r, x, t) = \mathrm{Re}\{R(r)\,\varphi(x, t)\exp[i(k_0 x - \omega_0 t)]\}$$

where $R(r)$ is a radial eigenfunction as a result of separation of variables of the wave equation [1, 2, 4], $\varphi(x, t)$ is the modulation of the monochromatic wave $\exp[i(k_0x - \omega_0 t)]$, and $\varphi(x, t)$ satisfies a fCNLS equation:

$$i(\varphi_t + \omega_0'\varphi_x) + \tfrac{1}{2}\omega_0''\varphi_{xx} + \frac{\omega_0 n_2}{2n_0}|\varphi|^2\varphi = F(x, t). \tag{1}$$

$F(x, t)$ is the external disturbance added to the CNLS equation. In this article, this disturbance is assumed to be a slowly oscillating wave traveling at the group velocity of the pulse. Explicitly,

$$F(x, t) = f(x - v_g t)\exp[i(kx - \Omega t)]. \tag{2}$$

Let

$$\varphi(x, t) = v(x, t)\exp[i(kx - \Omega t)]. \tag{3}$$

Substituting (2) and (3) into (1), we have

$$\left(\Omega - \omega_0' k - \frac{\omega_0'' k^2}{2}\right)v + \frac{\omega_0''}{2}v_{xx} + \frac{\omega_0 n_2}{2n_0}|v|^2 v + i(v_t + v_g v_x) = f(x - v_g t). \tag{4}$$

Equation (4) has a solution of the form

$$v(x, t) = u(\xi), \qquad \xi = x - v_g t \tag{5}$$

with $u(\xi)$ satisfying

$$u'' - \mu^2 u + \frac{1}{\nu^2}u^3 = g(\xi), \tag{6}$$

where

$$\mu^2 = (2/\omega_0'')((\omega_0''/2)k^2 + \omega_0' k - \Omega) > 0 \qquad \text{(measure of the pulse power)}, \tag{7}$$

$$\nu^2 = (n_0\omega_0'')/(\omega_0 n_2), \tag{8}$$

$$g(\xi) = (2/\omega_0'')f(\xi). \tag{9}$$

It is assumed that g is of compact support (i.e., $g(\xi)$ is nonzero only in a bounded closed interval). We denote $\xi_- = \inf supp(g)$, $\xi_+ = \sup supp(g)$. The main result of this paper is that for a given ν and g, (6) possesses solutions only if μ is sufficiently large and (6) possesses no solutions if μ is sufficiently small. In the following we justify this claim.

Let $\mathscr{B}$ be a complete metric space defined as

$$\mathscr{B} = \{\varphi \mid \varphi \in C(\mathbb{R}), \|\varphi\| = \sup_{x\in\mathbb{R}} |\varphi(x)| \exp(\mu|x|) \leq M \text{ for a fixed } M > 0.\}$$

Theorem. *For a given M and ν, if μ is sufficiently large, then* (6) *admits at least one solitary solution.*

Proof. $u(\xi)$ as a solitary solution to (6) infers that $u(\pm\infty) = 0$. Then (6) can be converted into an integral equation

$$u(\xi) = \int_{-\infty}^{\infty} K(\xi, \tau)[g(\tau) - (1/\nu^2)u^3(\tau)]\, \mathrm{d}\tau \equiv T(u)(\xi) \tag{10}$$

where

$$K(\xi, \tau) = (1/2\mu)\exp(-\mu|\xi - \tau|).$$

It is readily shown that if

$$\frac{1}{2\mu}\left[\frac{M^2}{\nu^2\mu} + \frac{2\max\{\exp(2\mu|\xi_-|), \exp(2\mu|\xi_+|)\}}{M}\int_{\xi_-}^{\xi_+} \cos h(u\xi)|g(\xi)|\, \mathrm{d}\xi\right] \leq 1, \tag{11}$$

and

$$\frac{3M^2}{2\mu^3\nu^2} < 1, \tag{12}$$

then T defined by (10) is a contraction map in $\mathscr{B}$. (11)-(12) always hold as long as μ is sufficiently large, thus $u = T(u)$ has a unique solution in $\mathscr{B}$ if μ satisfies (11)-(12). Further, if $u \in \mathscr{B}$, then $T(u) \in C^2(\mathbb{R})$ by (10). Hence $u = T(u)$ is a classical solution of (6). This completes the proof. □

Remark. The above theorem only implies that there exists at least one solitary solution to equation (6). And it does not imply that this solitary solution is the only solution to equation (6). We cannot exclude the possibility that problem (6) possesses some solitary solutions which do not satisfy equations (11)-(12). Actually, problem (6) has multiple solutions. Next, we will prove that (6) possesses at least three solitary solutions, two of which are not in $\mathscr{B}$, if μ is sufficiently large. In fact numerically we have found four solitary solutions to (6). The solution confirmed in the above theorem is the one having smallest amplitude among others. This solution approaches the null solution as the disturbance $f(\xi)$ vanishes.

To show that at least three solutions exist, let $u = \varphi(\xi)$ be a solitary solution to (6). Note that φ must be bounded. Define

$$\Theta = \{\xi \mid \xi \in \mathbb{R}, \varphi(\xi) \neq 0, \varphi'(\xi) = 0\}.$$

By the theorem and $g(\xi)\not\equiv 0$, Θ is nonempty if μ is sufficiently large. Now let $z=\inf\Theta$ and $N=\varphi(z)$. Then $u=\varphi(\xi)$ is monotonic in $(-\infty, z]$, and hence φ has an inverse $\xi=\xi(\varphi)$ defined on $(0, N]$ with range $(-\infty, z]$.

Multiplying (6) by $u'(\xi)$ and integrating the resulting equation with respect to ξ from $-\infty$ to z, we have

$$\mu^2=\frac{1}{2\nu^2}N^2-\frac{2}{N^2}\int_0^N g(\xi(\varphi))\,\mathrm{d}\varphi. \tag{13}$$

The curve S in the μ-N-plane defined by (13) has continuous branches. Therefore, for $\mu=\mu_0\geqslant 0$, the number of solutions to (6) is the number of intersections of the line $\mu=\mu_0$ with all branches of S in the μ-N-plane.

Evaluating $\int_0^N g(\xi(\varphi))\,\mathrm{d}\varphi$ by the mean value theorem, one can see that $N=0$ and $N=\pm\sqrt{2}\,\nu\mu$ are three asymptotes to S as μ goes to infinity. Thus, (6) possesses three solitary solutions if μ is sufficiently large. The branch of S which has $N=0$ as its asymptote corresponds to the solution confirmed by the theorem, since small amplitude makes the contraction map possible and the contracting fixed point is unique.

By (13), it can be shown that $\mu_c^2=\min_{N\in\mathbb{R}}\mu^2>0$. Hence there exist no solitary solutions to (6) with $\mu^2<\mu_c^2$.

3. Numerical results

Numerical solutions are found by solving (6) analytically from $-\infty$ to ξ_- and solving an initial value problem on (ξ_-,∞) as follows

$$u''-\mu^2u+(1/\nu^2)u^3=g(\xi),\quad \xi_-<\xi<\infty, \tag{14}$$

$$u(\xi)=\sqrt{2}\,\mu\nu\,\mathrm{sech}(\mu(\xi-\xi_0)),\quad \xi\leqslant\xi_-, \tag{15}$$

$$u'(\xi)=-\sqrt{2}\,\mu^2\nu\,\mathrm{sech}(\mu(\xi-\xi_0))\tanh(\mu(\xi-\xi_0)),\quad \xi\leqslant\xi_- \tag{16}$$

where the phase shift ξ_0 will be determined by the following analysis.

With $u(-\infty)=u'(-\infty)=0$, the integration of the product of (6) with $u'(\xi)$ from $-\infty$ to $\zeta\geqslant\xi_+$ results in

$$\tfrac{1}{2}(u')^2(\zeta)=P(u)(\zeta)+B(\xi_0,\mu),\quad \zeta\geqslant\xi_+, \tag{17}$$

where

$$P(u)(\zeta)=(u^2(\zeta)/4)(2\mu^2-u^2(\zeta)/\nu^2), \tag{18}$$

$$B(\xi_0,\mu)=\int_{\xi_-}^{\xi_+} g(\xi)\,u'(\xi)\,\mathrm{d}\xi. \tag{19}$$

Fig. 1 shows the curve of $P(u)+B(\xi_0,\mu)$ vs. u. We can see that solitary solutions to (6) exist if and only if [6]

$$B(\xi_0,\mu)=0. \tag{20}$$

It is this condition which determines the phase shift ξ_0. To find ξ_0, we can solve (14)-(16) up to ξ_+ for a trial value of ξ_0 and compute B defined by (19). Using a do loop we can obtain a curve of B vs. ξ_0. The intersections of the curve with the ξ_0 axis are the solutions of (20). The multiplicity of the solutions of (20) is equal to the number of solitary solutions of (6).

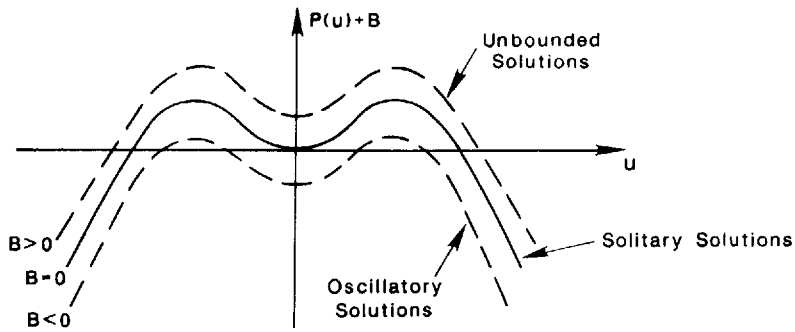

Fig. 1. Necessary and sufficient condition of existence of solitary solutions to (17). $P(u)$ and $B(\xi_0, \mu)$ are defined in (18)-(19).

As a numerical example, we take g as a Gaussian

$$g(\xi) = \begin{cases} \gamma(5/\sqrt{2\pi}) \exp(-12.5\,\xi^2), & |\xi| \leq 1, \\ 0, & |\xi| > 1, \end{cases}$$

and $\nu = 1.0$. As $\gamma = 1.0$ and $\mu = 2.0$, we find four values of ξ_0: $\xi_{01} = -1.5203$, $\xi_{02} = -0.0298$, $\xi_{03} = 1.5173$, $\xi_{04} = 0.0308$. The corresponding solutions are shown in Fig. 2.

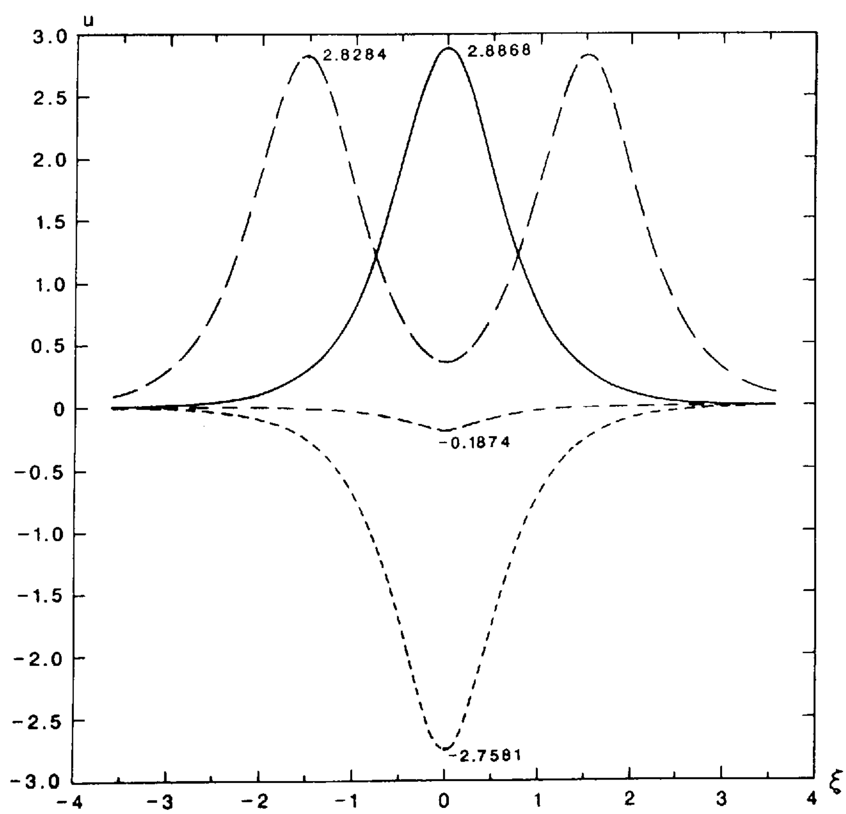

Fig. 2. Four solitary solutions to equation (6). Here we take $\nu = 1.0$, $\mu = 2.0$, $g(\xi) = \gamma(5/\sqrt{2\pi}) \exp(-12.5\,\xi^2)$, $|\xi| \leq 1$; 0, elsewhere. The correspondence of the solutions and phase shifts is: $\xi_{01} = -1.5203$ for u_1, $\xi_{02} = -0.0298$ for u_2, $\xi_{03} = 1.5173$ for u_3, $\xi_{04} = 0.0308$ for u_4.

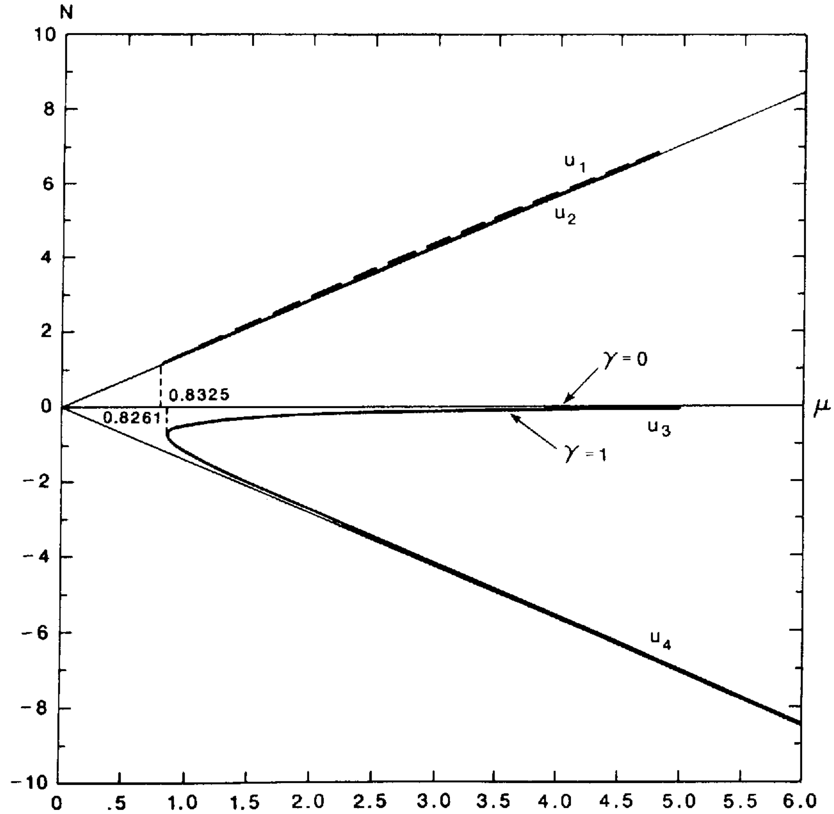

Fig. 3. μ–N diagram S determined by (13). The dashed curve corresponds to the solutions of two crests (see solution u_1 in Fig. 2). ν and g are the same as those in Fig. 2.

Fig. 3 shows the curve S for $\gamma = 1.0$ and $\nu = 1.0$. From Fig. 3, we see that a solitary solution exists to (6) only when μ is large enough (i.e., the supplying energy must be large enough). There exist four solutions, two solutions, and no solution for $\mu > 0.8325$, $0.8261 < \mu < 0.8325$, and $\mu < 0.8261$ respectively. Corresponding to the μ–N diagram of Fig. 3, the energy density $\int_{-\infty}^{\infty} u^2(\xi)\,\mathrm{d}\xi$ of the electric field vs. μ is shown in Fig. 4.

4. Concluding remarks

We have studied solitary pulse transmission in a fiber under a disturbance moving at the group velocity of the pulse. The pulse is the amplitude modulated near monochromatic plane wave $\exp[\mathrm{i}((k_0 + k)x - (\omega_0 + \Omega)t)]$. The amplitude modulation function $\varphi(x, t)$ satisfies a forced cubic-nonlinear Schrödinger equation. The forcing term $F(x, t)$ is due to the moving disturbance. The zeroth order approximation of the dispersion relation is not affected by the geometry of the cross-section of the fiber. The transmitted power, measured by μ^2 (see equation (7)), is a crucial parameter in our problem. A solitary pulse solution of equation (6) exists if and only if μ^2 is sufficiently large. Namely, for a solitary pulse to overcome the blocking, the intensity of the pulse must be sufficiently strong. This is the main conclusion of this paper.

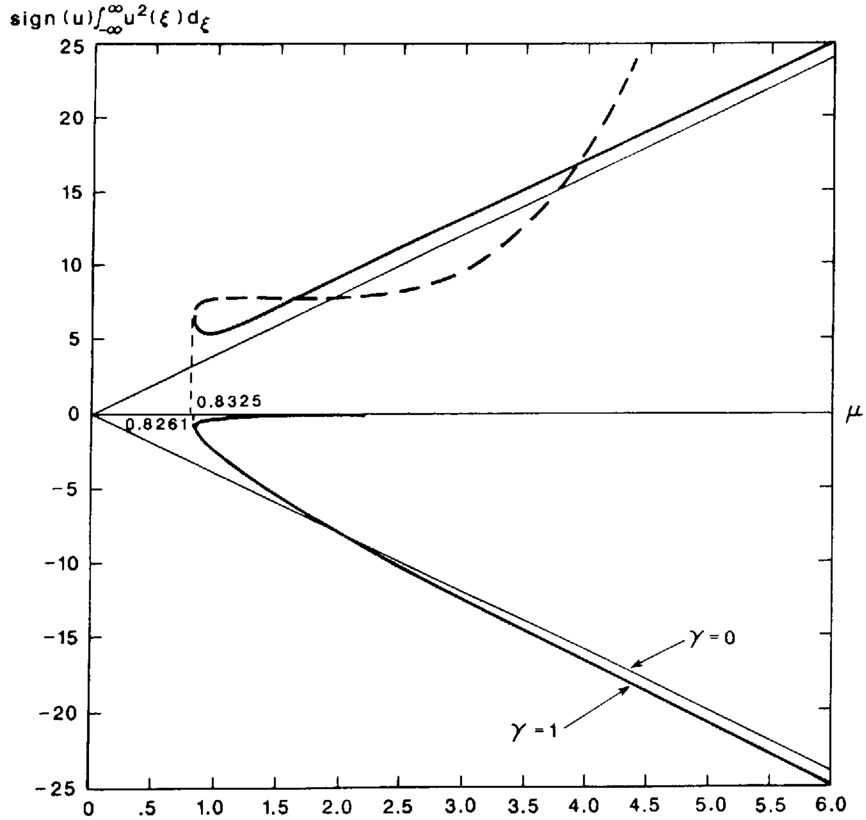

Fig. 4. Curves $\mathrm{sign}(u)\int_{-\infty}^{\infty} u^2(\xi)\,d\xi$ vs. μ. Here ν and g are the same as those in Fig. 2.

To my knowledge, there are no experimental results available on this type of blocking. External forcing usually comes into a system through boundary conditions. It might be perceivable to exert the disturbance in the medium outside of the fiber, in which the solitary pulses are transmitted. The technical details of how to exert such a disturbance remain to be investigated.

References

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