Solitary waves on a shelf

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By asymptotic analysis, it has been demonstrated that there exists a stationary solitary wave on the downstream side of a flat shelf when the upstream velocity U^* is greater than or equal to $U_C^* > U^{(0)^*}$, where $U^{(0)^*}$ is the propagation speed of shallow-water waves and U_C^* is determined by Eq. (12). The solitary wave is found by solving a forced Korteweg–de Vries equation. The amplitude of the solitary wave is proportional to the upstream velocity and the downstream elevation is inversely proportional to the upstream velocity. A second branch of solutions of the forced Korteweg–de Vries equation are also found, which are uniform flows both far upstream and far downstream.

In this Brief Communication, we announce our finding of the existence of a stationary solitary wave on a shelf when the upstream velocity U^* is greater than or equal to $U_C^* > U^{(0)^*}$, where $U^{(0)^*}$ is the propagation speed of shallow-water waves and U_C^* is determined by Eq. (12). When this condition prevails, the upstream flow is a solitary wave tail and the downstream flow is a complete solitary wave whose base is higher than that of the upstream solitary wave tail. There is a smooth transition region that connects the upstream solitary wave tail and the base of the downstream solitary wave (see Fig. 1).

The stationary solitary wave illustrated in Fig. 1 is different from a solitary wave that surges from an upstream deeper water zone to a downstream shelf and disintegrates into a train of smaller solitary waves, which is the soliton fission problem, as first studied by Madsen and Mei.[1] Our results are also different from those obtained by King and Bloor.[2] They studied the free surface flows over a step with the same fluid flow configuration as that described in the present work, except that they restricted themselves to only single-layer fluid flows. Instead of finding a solitary wave downstream, their results indicate that the downstream flow is uniform.

The existence of the solitary wave in this announcement, although not yet rigorously proved mathematically, can be intuitively justified. It is well known that at a supercritical speed there exists a stable solitary wave in each single-layer free surface flow. A bottom obstruction, such as a shelf, only alters the shape of the solitary wave, called the free solitary wave, in the flat channel but does not completely remove it. The altered solitary wave is considered to be a perturbation of the free solitary wave by the obstruction, as explained by Vanden-Broeck.[3] This explanation is supported by much evidence.[4] The solitary wave on a shelf in a two-layer flow in a closed channel is a perturbation of the interfacial free solitary wave. The existence of the free solitary wave was mathematically proved by Amick and Turner,[5] and was numerically justified by Turner and Vanden-Broeck.[6]

A second branch of solutions is the perturbation, by the shelf, of the unstable uniform flows in the case where there is no bottom obstruction. This perturbation is a solution that is uniform both far upstream and far downstream. Solutions on this branch are supposedly unstable, yet the justification of this instability claim seems not trivial and is deferred to subsequent research. The solutions found by King and Bloor[2] may be considered as the solutions on this branch.

Both of the aforementioned two branches of solutions have been found as solutions of a stationary forced Korteweg–de Vries equation (sfKdV). This equation was derived as the first-order asymptotic approximation of the free surface or interface of fluid flows over an obstruction. Hence, the sfKdV is a model equation for our problem. Next, we describe the meaning of this sfKdV.

Consider fluid flows in a two-dimensional channel. The bottom of the channel has a shelf and is otherwise flat. The transition zone from the upstream flat bottom to the downstream flat shelf is so short that when considering long waves, the transition is regarded as a step jump.

If one considers stratified fluid flows in an open channel, the first-order approximation of both the free surface and interface yields sfKdV equations. Now, as an example, we consider two-layer fluid flows in a closed channel: a bottom fluid of density ρ_- and depth H_- and a lighter top fluid of density ρ_+ (i.e., $\rho_+ < \rho_-$) and depth H_+. The two fluids are confined in a closed channel by a horizontal lid above and a semi-infinite step below with the step height $S(<H_-)$ (see Fig. 2). The flow is assumed to be two dimensional, stationary, and irrotational in each layer. Let the x^* axis be aligned along the longitudinal direction, and the y^* axis vertically opposite to the gravitational direction. The subscripts "±" signify the quantities of the upper layer and the lower layer fluid, respectively. The flow potential functions are $\Phi_\pm^*$. The upstream uniform velocities are $U_\pm^*$. The interfacial profile is $y^* = H_- + \eta^*(x^*)$. The gravitational acceleration is g. The upper boundary of the channel is $y^* = H_- + H_+$ and the lower boundary is $y^* = S\mathscr{H}(x^*)$, where $\mathscr{H}(x^*)$ is the Heaviside step function defined by

$$\mathscr{H}(x^*) = \begin{cases} 1, & \text{when } x^* > 0, \\ 0, & \text{otherwise,} \end{cases}$$

whose derivative is the Dirac delta function $\delta(x^*)$.

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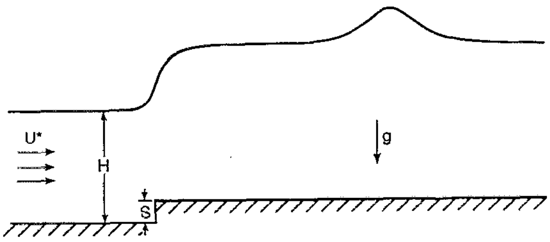

FIG. 1. The solitary wave on a shelf.

The first-order approximation of the interface profile yields a sfKdV:

$$m_1\eta_x^{(1)}+m_2\eta^{(1)}\eta_x^{(1)}+m_3\eta_{xxx}^{(1)}=m_4\delta(x), \tag{1}$$

where the coefficients m_k $(k=1,2,3,4)$ are given by

$$m_1=2\left(\frac{\rho}{\sigma}\lambda_+U_+^{(0)}+\lambda_-U_-^{(0)}\right), \tag{2}$$

$$m_2=3\left(\frac{\rho}{\sigma^2}U_+^{(0)^2}-U_-^{(0)^2}\right), \tag{3}$$

$$m_3=-\tfrac{1}{3}(\sigma\rho U_+^{(0)^2}+U_-^{(0)^2}), \tag{4}$$

$$m_4=U_-^{(0)^2}, \tag{5}$$

and

$\epsilon=(H_-/L)^2\ll 1,\quad \epsilon^2=S/H_-$ (small step assumption),

$\sigma=H_+/H_-,\quad \rho=\rho_+/\rho_-,\quad \gamma=U_+^*/U_-^*,$

$U_\pm=U_\pm^*/(gH_-)^{1/2}$

$=U_\pm^{(0)}+\epsilon\lambda_\pm$ (near critical velocities),

$(x,y)=(\epsilon^{1/2}x^*,y^*)/H_-$ (long wave assumption),

$\eta=\eta^*/H_-=\epsilon\eta^{(1)}+O(\epsilon^2),$

where * signifies quantities with dimension. The critical velocities $U_\pm^{(0)}$ are determined by

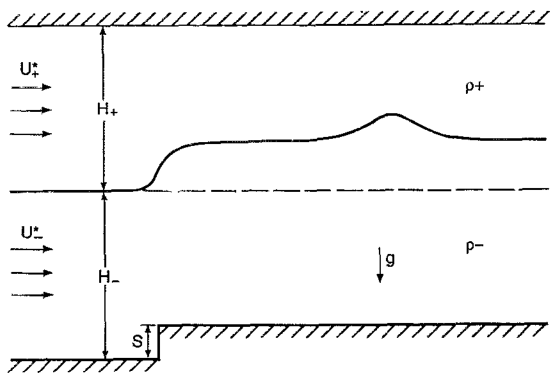

FIG. 2. The internal wave of a two-layer fluid over a shelf in a closed channel.

$$U_-^{(0)^2}=\sigma(1-\rho)/(\sigma+\gamma^2\rho),\quad U_+^{(0)}=\gamma U_-^{(0)}. \tag{6}$$

We use the sfKdV (1) as our model equation, which is an accurate model equation for near critical flows over a semicircular bump for a large range of ϵ values (such as $0\leqslant\epsilon<0.7$)[4], but omit its derivation in this work. In the following, when $\gamma=1$, we present two branches of solutions of boundary-value problems (BVP) for the sfKdV (1):

$$\lambda\eta_x^{(1)}+2\alpha\eta^{(1)}\eta_x^{(1)}+\beta\eta_{xxx}^{(1)}=P\delta(x),\quad -\infty<x<\infty, \tag{7}$$

$$\eta^{(1)}(-\infty)=0,\quad \eta^{(1)}(\infty)=a, \tag{8}$$

$$\eta_x^{(1)}(\pm\infty)=\eta_{xx}^{(1)}(\pm\infty)=0. \tag{9}$$

In the above, $\alpha=m_2/(2\bar{m}_1)<0$, $\beta=m_3/\bar{m}_1<0$, and $P=m_4/\bar{m}_1>0$ are determined by the density ratio and the depth ratio of the two fluids, $\bar{m}_1=m_1/\lambda$, and $\lambda>0$ signifies supercritical flows, and since the far downstream interface is an elevation, a is positive. Integrating the sfKdV (7) from $-\infty$ to ∞ with respect to x, we obtain a relationship among P, λ, and a:

$$\lambda a+\alpha a^2=P. \tag{10}$$

We can solve this equation to obtain two solutions for a,

$$a_\pm=[-\lambda\pm(\lambda^2+4\alpha P)^{1/2}]/2\alpha, \tag{11}$$

when

$$\lambda\geqslant 2(-\alpha P)^{1/2}\equiv\lambda_C=(U_C^*-U^{(0)^*})/\epsilon(gH_-)^{1/2}. \tag{12}$$

Here, $U_C^*=(U^{(0)}+\epsilon\lambda_C)(gH_-)^{1/2}$ and $U^{(0)^*}=U^{(0)}\times(gH_-)^{1/2}$.

Integrating twice the BVP (7)–(9) results in an initial value problem (IVP)

$$(3\beta/2\alpha)(\eta_x^{(1)})^2=Q_\pm(\eta^{(1)}),\quad x>0, \tag{13}$$

$$\eta^{(1)}(0)=A_\pm, \tag{14}$$

where

$$Q_\pm(\eta^{(1)})=(W_\pm-\eta^{(1)})(\eta^{(1)}-a_\pm)^2, \tag{15}$$

$$A_\pm=a_\pm(4P-\lambda a_\pm)/6P \tag{16}$$

$$W_\pm=-\frac{\lambda}{2\alpha}\left[1\pm2\left(1+\frac{4\alpha P}{\lambda^2}\right)^{1/2}\right]. \tag{17}$$

Since $Q_-(\eta^{(1)})<0$ when $W_-<\eta^{(1)}<a_-$, the IVP (13) and (14) does not have a bounded solution. Thus, we take only a_+ in (11) to search for the two branches of solutions.

Solitary wave branch. The solitary wave on the shelf is

$$\eta^{(1)}(x)=a_++(W_+-a_+)\times\operatorname{sech}^2\left[\left(\frac{\alpha(W_+-a_+)}{6\beta}\right)^{1/2}(x-x_0^+)\right]. \tag{18}$$

It seems difficult to determine the phase shift x_0^+ analytically since Eq. (13) becomes singular when $\eta^{(1)}=a_+$. Thus, we used the ordinary differential equation (ODE)

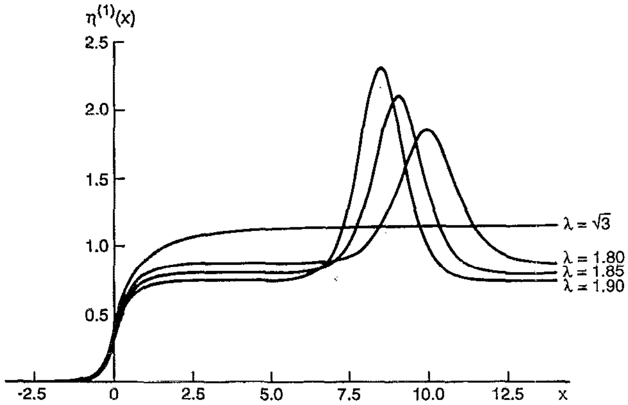

FIG. 3. Solitary wave solutions of (7)–(9) with $\alpha=-3/4$, $\beta=-1/6$, and $P=1$ for $\lambda=\sqrt{3}$, 1.80, 1.85, and 1.90.

solver in MATHEMATICA called NDSolve to obtain our solitary wave solutions. Some solutions are shown in Fig. 3.

Uniform flow branch. The analytic expression for the solutions on this branch is

$$\eta^{(1)}(x)=-\frac{3\lambda}{2\alpha}\operatorname{sech}^2\left\{\left(\frac{\lambda}{-4\beta}\right)^{1/2}\times\left[x-\left(\frac{4\beta}{-\lambda}\right)^{1/2}\operatorname{arcsech}\left(-\frac{2\alpha}{3\lambda}A_+\right)^{1/2}\right]\right\} \tag{19}$$

when $x<0$ and

$$\eta^{(1)}(x)=a_+ +(a_+ -W_+)\operatorname{csch}^2\left[\left(\frac{\alpha(W_+ -a_+)}{6\beta}\right)^{1/2}x+\operatorname{arccoth}\left(\frac{a_+ -A_+}{W_+ -a_+}\right)^{1/2}\right] \tag{20}$$

when $x>0$. Graphics of solutions on this branch are shown in Fig. 4 which qualitatively agree with numerical solutions shown in Fig. 2 in Ref. 2. In terms of quantity, when the upstream Froude number is 2 and the step height is $0.2H$ for a single-layer fluid flow, the downstream free surface height is $1.27H$ according to Ref. 4 and $1.11H$ according to the present sfKdV theory. Although the sfKdV asymptotic approximation systematically underestimates the downstream elevation, it successfully demonstrates the existence of two branches of solutions: a shelf solitary wave and a shelf uniform flow.

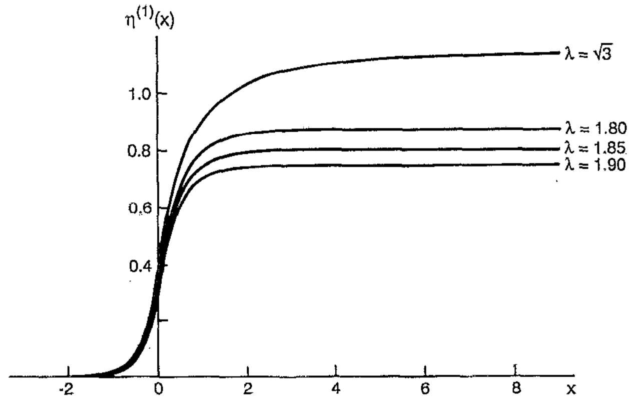

FIG. 4. Uniform flow solutions of (7)–(9) with $\alpha=-3/4$, $\beta=-1/6$, and $P=1$ for $\lambda=\sqrt{3}$, 1.80, 1.85, and 1.90.

ACKNOWLEDGMENT

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[1] O. S. Madsen and C. C. Mei, "The transformation of a solitary wave over an uneven bottom," J. Fluid Mech. **39**, 781 (1969).
[2] A. C. King and M. I. G. Bloor, "Free-surface flow over a step," J. Fluid Mech. **182**, 193 (1987).
[3] J.-M. Vanden-Broeck, "Free-surface flow over an obstruction in a channel," Phys. Fluids **30**, 2315 (1987).
[4] S. S. P. Shen, "Forced solitary waves and hydraulic falls in two-layer flows," J. Fluid Mech. **234**, 583 (1992).
[5] C. J. Amick and R. E. L. Turner, "A global theory of internal solitary waves in two-fluid systems," Trans. Am. Math. Soc. **298**, 431 (1986).
[6] R. E. L. Turner and J.-M. Vanden-Broeck, "Broadening of interfacial solitary waves," Phys. Fluids **31**, 2486 (1988).