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A simple derivation of rigid-rotation formula

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Abstract A brief, easy and elementary way of deriving the vector formula for the finite rigid-body rotation about a fixed axis is presented in this article.

Zusammenfassung Dieser Artikel giebt eine einfache, kurze und elementare Methode zur Herleitung der Vektorformel fur die rendliche Rotation eines starren Koerpers um eine feste Achse.

The vector description of a rigid body rotating around an arbitrary fixed axis is a classical problem that has been studied repeatedly in recent years, as pointed out by Beatty (1977). Of course, the published articles have their own styles and use a variety of methods. For example, Grubin (1962) got his result by solving a vector differential equation; Beatty (1963) used the method of vector analytic geometry; Pearlman (1967) gave a similar geometrical construction; Palazzolo (1976) obtained the result through a very complicated process of matrix algebra that later was improved by Neuberger (1977) but there is no intrinsic difference between their methods, and I doubt that either may be easily understood by an undergraduate student. Pearlman's method appears to be the most elementary; nevertheless, his procedure of vector projection tends to make the analysis obscure. One may find detailed historical materials with numerous references about this problem in the review article by Beatty (1977).

This article presents a brief, easy and elementary derivation of the vector formula for the finite rigid-body rotation about a fixed axis. The method is similar to Pearlman's in the choice of reference frame, but new in its use of the matrix formula for the rotation vector in terms of the initial directed vector of a body point. The matrix formula is constructed by the elementary methods of geometry and the analytical representation of a vector in terms of its components.

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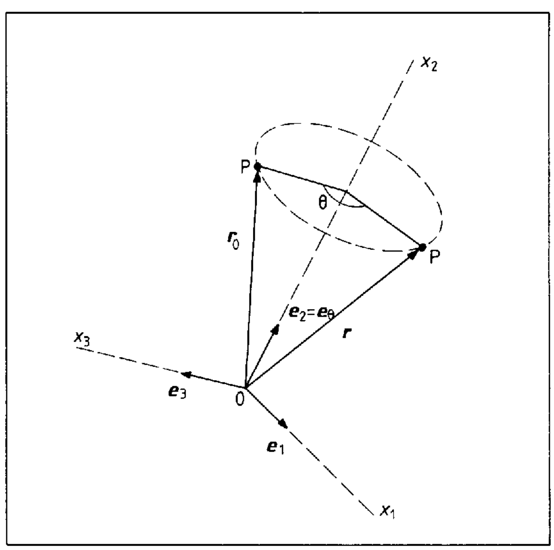
Let a unit vector $\boldsymbol{e}_\theta$ be along the fixed axis of rotation, and let a reference frame $0\text{-}x_1x_2x_3$ be fixed in space with its x_2 direction along $\boldsymbol{e}_\theta$, as shown in figure 1.

After the rotation through an angle θ in the usual right-handed sense around $\boldsymbol{e}_\theta$, the position vector $\boldsymbol{r}_0$ of a particle P becomes $\boldsymbol{r}$. We assume that both vectors are referred to the same fixed frame. It may be seen from the geometry in figure 2 that:

$$\left.\begin{aligned} x_1 &= x_{10}\cos\theta + x_{30}\sin\theta \\ x_2 &= x_{20} \\ x_3 &= -x_{10}\sin\theta + x_{30}\cos\theta \end{aligned}\right\} \qquad (1)$$

Figure 1 A rigid body rotating around a fixed axis.



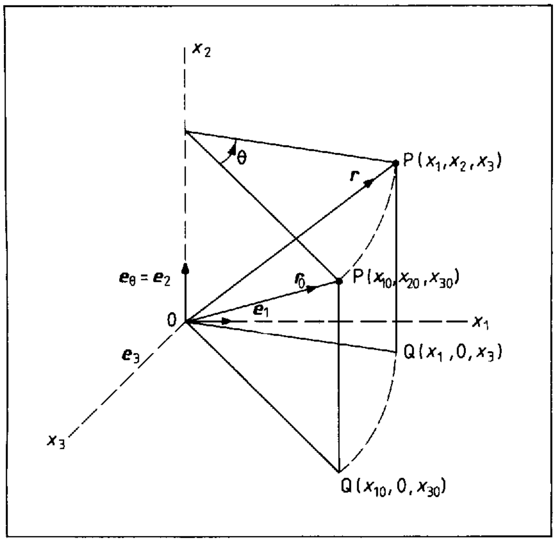


Figure 2 Analysis of the rotation.

where $x_{i0} = \boldsymbol{r}_0 \cdot \boldsymbol{e}_i$, $x_i = \boldsymbol{r} \cdot \boldsymbol{e}_i$ $(i = 1, 2, 3)$ define the cartesian coordinates of P at $\boldsymbol{r}_0$ and $\boldsymbol{r}$ respectively, and $\boldsymbol{e}_i$ denotes the basic vectors of the frame with $\boldsymbol{e}_2 = \boldsymbol{e}_\theta$.

If we note

$$\mathbf{A} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\boldsymbol{r}_0 = \sum_1^3 (\boldsymbol{r}_0 \cdot \boldsymbol{e}_i)\boldsymbol{e}_i, \qquad \boldsymbol{r} = \sum_1^3 (\boldsymbol{r} \cdot \boldsymbol{e}_i)\boldsymbol{e}_i \tag{2}$$

then the three scalar equations (1) may be written

$$\boldsymbol{r} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)\mathbf{A}\begin{pmatrix} \boldsymbol{e}_1 \cdot \boldsymbol{r}_0 \\ \boldsymbol{e}_2 \cdot \boldsymbol{r}_0 \\ \boldsymbol{e}_3 \cdot \boldsymbol{r}_0 \end{pmatrix}$$

$$= (\boldsymbol{e}_2 \cdot \boldsymbol{r}_0)\boldsymbol{e}_2 + \sin\theta[(\boldsymbol{e}_3 \cdot \boldsymbol{r}_0)\boldsymbol{e}_1 - (\boldsymbol{e}_1 \cdot \boldsymbol{r}_0)\boldsymbol{e}_3]$$
$$+ \cos\theta[(\boldsymbol{e}_1 \cdot \boldsymbol{r}_0)\boldsymbol{e}_1 + (\boldsymbol{e}_3 \cdot \boldsymbol{r}_0)\boldsymbol{e}_3].$$

We now recall the formula

$$\boldsymbol{e}_1(\boldsymbol{e}_3 \cdot \boldsymbol{r}_0) - \boldsymbol{e}_3(\boldsymbol{e}_1 \cdot \boldsymbol{r}_0) = (\boldsymbol{e}_3 \times \boldsymbol{e}_1) \times \boldsymbol{r}_0$$

and equation (2) for $\boldsymbol{r}_0$ to obtain

$$\boldsymbol{r} = (\boldsymbol{e}_\theta \cdot \boldsymbol{r}_0)\boldsymbol{e}_\theta + \sin\theta(\boldsymbol{e}_\theta \times \boldsymbol{r}_0) + \cos\theta[\boldsymbol{r}_0 - (\boldsymbol{e}_\theta \cdot \boldsymbol{r}_0)\boldsymbol{e}_\theta] \tag{3}$$

which is the desired result.

We point out, incidentally, that the infinitesimal rotation may be readily obtained from here. In this case, $\theta \ll 1$, $\sin\theta \approx \theta$, $\cos\theta \approx 1$. It follows that

$$\Delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_0 = (\theta\boldsymbol{e}_\theta) \times \boldsymbol{r}_0$$

i.e., with $\boldsymbol{\Delta\theta} = \theta\boldsymbol{e}_\theta$, $\Delta\boldsymbol{r} = \boldsymbol{\Delta\theta} \times \boldsymbol{r}$.

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