



A Quantitative Study on the Stability of Quadratic Delay Difference Systems

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Abstract—This paper estimates the size of the stability region around zero for quadratic delay difference systems. When the initial disturbance is in the asymptotic stability region, the solution of the initial value problem of the quadratic delay difference system approaches zero. Examples are given for a three-dimensional system and three one-dimensional equations to demonstrate both stability and instability. Examples 2–4 show that when parameters in the systems do not satisfy the stability conditions, the zero solutions can be unstable. Three evolution features of initial disturbances are shown numerically: decaying to zero, being amplified but bounded, and growing to infinity. Example 3 further shows that the stable zero solution may not be a global attractor. Numerical results confirm the conclusions of the main theorem in this paper and imply that our estimation of the size of the stability region are of reasonable accuracy. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Quadratic delay difference systems, Uniform stability, Uniform asymptotic stability, Stability region, Asymptotic stability region.

1. INTRODUCTION

When considering a dynamical system, it is often a question whether an equilibrium point is stable. The same is true for delay difference systems. After a coordinate transform, the stability analysis for any equilibrium point can be transformed to the stability problem of the zero solution. The purpose of this paper is to study stability properties of the zero solution of difference delay systems.

Stability studies may be classified into two categories. The first one is qualitative stability study which assures that zero solutions are in principle stable or unstable. Elaydi and Zhang [1] is an example of such a qualitative stability study and presents some stability criteria for the finite delay difference systems of general form in terms of the discrete Liapunov functionals and Liapunov functions. The analysis method in [1] and in the present paper is well summarized in [2], where one can find an extensive treatment of the stability theory of difference equations

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without delay. Zhang [3] extended the results in [1] to the difference systems of infinite delay. A relaxed stability condition was found in [4], which is an improvement of the results in [1,3]. Despite the above progress in the qualitative stability research, in practical applications, one often needs to know what is the maximal tolerance of the perturbation from an equilibrium such that the perturbation can still be attracted back to the equilibrium. This requires more careful estimation of the functions involved in the proof of the above qualitative stability results and numerical simulations for various initial disturbances. We come to the second category of stability studies: quantitative description of the size of the stability region of the zero solutions and numerical simulation of the evolutions of the initial perturbations from an equilibrium.

In this paper, we will concretely describe the size of the stability region so that as long as the initial disturbances (i.e., initial data) are restricted within this region, the desired uniformly stable and/or uniformly asymptotically stable properties are guaranteed. Numerical simulations are described for several different systems and three types of evolutions of the initial perturbations are shown: stable evolution, unstable but bounded evolution, and unstable and unbounded evolution. To our knowledge, both the analytic and numerical results included in the this paper are new and they are the first presentation on the quantitative description of the sizes of the stability region and asymptotic stability of the zero solution of quadratic delay difference systems.

The context of this paper is arranged as follows. The preparation materials for describing our main stability result of this paper are in the first part of Section 2, and Theorem 2 as the main result is stated in the second part of Section 2. The proof of the main result is in Section 3. Four numerical examples are described in Section 4, and Section 5 includes conclusions and discussions.

2. MAIN RESULTS

To describe the main result of this paper, we include some preliminary knowledge on the stability of the delay difference systems. For delay difference systems of the following general form:

$$x(n+1) = f(n, x_n), \quad n \in Z^+, \quad (1)$$

where Z^+ denotes the set of nonnegative integers $x \in R^k$ (k -dimensional Euclidean space) and $x_n(s) = x(n+s)$, for $s = -r, -r+1, \dots, -1, 0$ with some positive integer $r > 0$. Assume $f(n, 0) = 0$ for $n \in Z^+$, so that (1) always has the zero solution $x(n) = 0$. Clearly, for any given $n_0 \in Z^+$ and a given initial function

$$\varphi : \{-r, -r+1, \dots, -1, 0\} \rightarrow R^k,$$

there is a unique solution of (1), denoted by $x(n, n_0, \varphi)$, which satisfies (1) for all integers $n \geq n_0$ and

$$x(n_0 + s, n_0, \varphi) = \varphi(s), \quad \text{for } s = -r, -r+1, \dots, -1, 0.$$

Let

$$\|\varphi\| = \sup \{|\varphi(s)| : s \in \{-r, -r+1, \dots, -1, 0\}\}.$$

In the sequel, we will always assume that the variables n , s , i , and j take integer values and all the intervals and inequalities are discrete.

The following definitions and theorems from [1] will be used in the statement and proof of the main result.

DEFINITION 1. *The zero solution of (1) is Uniformly Stable (US) if for each $\varepsilon > 0$, and any $n_0 \in Z^+$, there exists a $\delta(\varepsilon) > 0$ independent of n_0 such that if $\|\varphi\| < \delta$, then*

$$|x(n, n_0, \varphi)| < \varepsilon, \quad \text{for all } n \geq n_0.$$

DEFINITION 2. The zero solution of (1) is Uniformly Asymptotically Stable (UAS) if it is US and there is a $\delta_0 > 0$ such that for each $\gamma > 0$, there exists an integer $N(\gamma) > 0$ independent of n_0 such that if $n_0 \in \mathbb{Z}^+$ and $\|\varphi\| < \delta_0$, then

$$|x(n, n_0, \varphi)| < \gamma, \quad \text{for all } n \geq n_0 + N(\gamma).$$

DEFINITION 3. A strictly increasing continuous function $W : [0, \infty) \rightarrow [0, \infty)$, with $W(0) = 0$, $W(u) > 0$ if $u > 0$ is called a wedge.

DEFINITION 4. The region Ω defined as

$$\Omega = \left\{ \varphi : \{-r, -r+1, \dots, -1, 0\} \rightarrow \mathbb{R}^k \mid \lim_{n \rightarrow \infty} x(n, n_0, \varphi) = 0 \right\}$$

is said to be the asymptotic stability region of the zero solution of (1).

THEOREM 1. Suppose there exists a Liapunov function $V : \mathbb{Z}^+ \times S_h \rightarrow [0, \infty)$, where $S_h = \{x \in \mathbb{R}^k : |x| < h\}$, such that

- (i) $W_1(|x|) \leq V(n, x) \leq W_2(|x|)$, and
- (ii) $\Delta V(n, x(n)) \leq -W_3(|x(n)|)$, when

$$P[V(n+1, x(n+1))] > V(s, x(s)), \quad \text{for } n-r \leq s \leq n.$$

Here W_i , $1 \leq i \leq 3$, are wedges, $P : [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $P(u) > u$ when $u > 0$, and

$$\Delta V(n, x(n)) \equiv V(n+1, x(n+1)) - V(n, x(n)) = V(n+1, f(n, x_n)) - V(n, x(n)),$$

with $x(n)$ being a solution of (1). Then the zero solution of (1) is UAS.

In this paper, we study the following quadratic delay difference systems:

$$x(n+1) = A_0 x(n) + A_1 x(n - \tau_1(n)) + X(n - \tau_2(n)) B_1 x(n), \quad (2)$$

where $n \in \mathbb{Z}^+$, $x \in \mathbb{R}^k$, A_0, A_1 are $k \times k$ constant matrices, and $X(n)$ and B_1^\top (which is the transpose of B_1) are $k \times k^2$ matrices

$$X(n) = [X_1(n), X_2(n), \dots, X_k(n)], \quad B_1^\top = [B_{11}, B_{21}, \dots, B_{k1}].$$

Here $X_i(n)$ is a matrix whose i^{th} row is $x^\top(n) = (x_1(n), x_2(n), \dots, x_k(n))$ and the other elements are all zero, i.e.,

$$X_i(n) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ x_1(n) & x_2(n) & \dots & x_k(n) \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

and

$$B_{i1} = \begin{bmatrix} b_{i1}^{11} & b_{i1}^{12} & \dots & b_{i1}^{1k} \\ b_{i1}^{12} & b_{i1}^{22} & \dots & b_{i1}^{2k} \\ \dots & \dots & \dots & \dots \\ b_{i1}^{1k} & b_{i1}^{2k} & \dots & b_{i1}^{kk} \end{bmatrix},$$

where $i = 1, 2, \dots, k$ and $\tau_j : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ with $0 \leq \tau_j(n) \leq r$, for some positive integer r ($j = 1, 2$).

For a vector x , its Euclidean norm is defined as

$$|x| = \left\{ \sum_{i=1}^k x_i^2 \right\}^{1/2},$$

and for a matrix A , its spectral norm is defined as

$$|A| = \{\lambda_{\max}(A^T A)\}^{1/2},$$

where and in the sequel, $\lambda_{\max}(\cdot)$ is the largest eigenvalue of the corresponding matrix and $\lambda_{\min}(\cdot)$ is the smallest eigenvalue. With the above choice of norms, it follows that $|X(n)| = |x(n)|$.

It is known from [5] that if the moduli of all eigenvalues of A_0 are less than one (in this case we say that A_0 is stable), then for any given positive definite symmetric matrix C , there exists a unique positive definite symmetric matrix H such that

$$C = H - A_0^T H A_0. \quad (3)$$

We take the quadratic form

$$V(x) = x^T H x$$

as the Liapunov function, where H is the solution of (3). Then there holds

$$\lambda_{\min}(H)|x|^2 \leq V(x) \leq \lambda_{\max}(H)|x|^2.$$

Hence, $V(n, x) = V(x) = x^T H x$ clearly satisfies Condition (i) in Theorem 1.

With the above preparation we can now state our main result as a theorem.

THEOREM 2. *Assume that A_0 is stable and there holds the following two conditions:*

- (i) $1 - \varphi(H)|A_1| > 0$, where $\varphi(H) = \sqrt{\lambda_{\max}(H)/\lambda_{\min}(H)}$, and
- (ii) $P \equiv \lambda_{\min}(C) - 2\mu\varphi(H)|A_0^T H A_1| - \mu^2\varphi^2(H)\lambda_{\max}(H)|A_1|^2 > 0$, where

$$\mu = \frac{|A_0|}{1 - \varphi(H)(|A_1| + \varepsilon_0|B_1|)},$$

with some constant $\varepsilon_0 : 0 < \varepsilon_0 < 1$ and $1 - \varphi(H)(|A_1| + \varepsilon_0|B_1|) > 0$.

Then we have the following conclusions:

- (a) the zero solution of (2) is UAS for arbitrary $r > 0$;
- (b) no solution $x(n, n_0, \varphi)$ leaves the ball $S_\varepsilon = \{x : |x(n, n_0, \varphi)| < \varepsilon\}$, for all $n \geq n_0$, whenever $\|\varphi\| < \delta(\varepsilon)$ with

$$\delta(\varepsilon) = \frac{\min\{(P/Q), \varepsilon\}}{\varphi(H)}, \quad (4)$$

where P as in (ii),

$$Q \equiv 2\mu\varphi(H)|B_1||H A_0| + 2\mu^2\varphi^2(H)|B_1||H A_1| + \mu^2\varphi^2(H)\lambda_{\max}(H)|B_1|^2;$$

and

- (c) the asymptotic stability region Ω contains at least a ball S_R with the radius

$$R = \frac{\min\{(P_0 - \alpha)/Q_0, \varepsilon_0\}}{\varphi(H)},$$

where P_0, Q_0, ε_0 , and α will be specified later.

In the above theorem, three conclusions are given. Conclusion (a) claims that the zero solution is asymptotically stable. This is a qualitative result. Conclusions (b) and (c) give the sizes of the uniform stability region and uniform asymptotic stability regions, respectively. They are quantitative results. A proof of the theorem is given in the next section.

3. PROOF AND EXTENSIONS OF THE MAIN RESULT

The proof of the theorem is divided into two steps. The first step is to show that we can construct a Liapunov function which is bounded from above, and hence, validates Conclusion (a) in the above theorem. The second step affirms Conclusions (b) and (c).

PROOF.

STEP I. For any given $\varepsilon > 0$ ($\varepsilon \leq \varepsilon_0$), we have

$$1 - \varphi(H) (|A_1| + \varepsilon|B_1|) \geq 1 - \varphi(H) (|A_1| + \varepsilon_0|B_1|) > 0. \quad (5)$$

Now choose $\delta(\varepsilon)$ as in (4). Let $n_0 \in Z^+$, $\|\varphi\| < \delta$, and $x(n) = x(n, n_0, \varphi)$. Then it follows from $|x(n)| < \delta$ for $n_0 - r \leq n \leq n_0$ that

$$V(x(n)) \leq \lambda_{\max}(H)|x(n)|^2 < \lambda_{\max}(H)\delta^2, \quad \text{for } n_0 - r \leq n \leq n_0.$$

We claim that

$$V(x(n)) < \lambda_{\max}(H)\delta^2, \quad \text{for all } n \geq n_0. \quad (6)$$

Suppose this is not true. Then there exists some integer $n^* \geq n_0$ such that

$$V(x(n)) < \lambda_{\max}(H)\delta^2, \quad \text{for } n_0 - r \leq n \leq n^* \text{ and } V(x(n^* + 1)) \geq \lambda_{\max}(H)\delta^2.$$

The bounds of $V(x)$ can be further written as

$$\lambda_{\min}(H)|x(n)|^2 \leq V(x(n)) < \lambda_{\max}(H)\delta^2 \leq \frac{\lambda_{\max}(H)\varepsilon^2}{\varphi^2(H)}, \quad \text{for } n_0 - r \leq n \leq n^*.$$

This implies that

$$|x(n)| < \varepsilon \leq 1, \quad \text{for } n_0 - r \leq n \leq n^*. \quad (7)$$

In addition,

$$\lambda_{\min}(H)|x(n)|^2 \leq V(x(n)) < \lambda_{\max}(H)\delta^2 \leq V(x(n^* + 1)) \leq \lambda_{\max}(H)|x(n^* + 1)|^2, \\ \text{for } n_0 - r \leq n \leq n^*,$$

implies that

$$|x(n)| < \varphi(H)|x(n^* + 1)|, \quad \text{for } n_0 - r \leq n \leq n^*. \quad (8)$$

From (2), we have

$$|x(n^* + 1)| \leq |A_0||x(n^*)| + |A_1|\varphi(H)|x(n^* + 1)| + \varepsilon|B_1|\varphi(H)|x(n^* + 1)|. \quad (9)$$

This implies that

$$|x(n^* + 1)| \leq \frac{|A_0|}{1 - \varphi(H)(|A_1| + \varepsilon|B_1|)} |x(n^*)| \leq \mu |x(n^*)|, \quad (10)$$

in virtue of (5).

By definition and equations (7), (8), and (10), we now have

$$\begin{aligned}
\Delta V(x(n^*)) &= V(x(n^* + 1)) - V(x(n^*)) \\
&= [x^\top(n^*)A_0^\top + x^\top(n^* - \tau_1(n^*))A_1^\top + x^\top(n^*)B_1^\top X^\top(n^* - \tau_2(n^*))] H \\
&\quad [A_0x(n^*) + A_1x(n^* - \tau_1(n^*)) + X(n^* - \tau_2(n^*))B_1x(n^*)] - x^\top(n^*)Hx(n^*) \\
&\leq -\lambda_{\min}(C)|x(n^*)|^2 + 2\varphi(H)[|A_0^\top HA_1| + |B_1||HA_0||x(n^*)| \\
&\quad + \varphi(H)|B_1||HA_1||x(n^* + 1)|] |x(n^*)||x(n^* + 1)| \\
&\quad + \lambda_{\max}(H)\varphi^2(H)[|A_1|^2 + |B_1|^2|x(n^*)|^2]|x(n^* + 1)|^2 \\
&\leq -[\lambda_{\min}(C) - 2\mu\varphi(H)|A_0^\top HA_1| - \mu^2\varphi^2(H)\lambda_{\max}(H)|A_1|^2]|x(n^*)|^2 \\
&\quad + [2\mu\varphi(H)|B_1||HA_0| + 2\mu^2\varphi^2(H)|B_1||HA_1| \\
&\quad + \varepsilon\mu^2\varphi^2(H)\lambda_{\max}(H)|B_1|^2]|x(n^*)|^3 \\
&\leq -P|x(n^*)|^2 + Q|x(n^*)|^3.
\end{aligned}$$

That is,

$$\Delta V(x(n^*)) \leq -(P - Q|x(n^*)|)|x(n^*)|^2. \quad (11)$$

From

$$\lambda_{\min}(H)|x(n)|^2 \leq V(x(n)) < \lambda_{\max}(H)\delta^2 \leq \lambda_{\max}(H)\frac{P^2}{Q^2\varphi^2(H)}, \quad \text{for } n_0 - r \leq n \leq n^*,$$

we have

$$|x(n)| < \frac{P}{Q}, \quad \text{for } n_0 - r \leq n \leq n^*.$$

In particular, we have

$$|x(n^*)| < \frac{P}{Q}. \quad (12)$$

Substitution of (12) into (11) yields

$$\Delta V(x(n^*)) \leq 0.$$

This contradicts the assumption $V(x(n^* + 1)) \geq \lambda_{\max}(H)\delta^2 > V(x(n^*))$.

Therefore, (6) holds, which implies that

$$|x(n)| < \varepsilon, \quad \text{for all } n \geq n_0.$$

Hence, the zero solution of (2) is US and the solution $x(n, n_0, \varphi)$ does not leave the ball S_ε , for all $n \geq n_0$, when $\|\varphi\| < \delta(\varepsilon)$ with $\delta(\varepsilon)$ as in (4).

STEP II. Next we assert that the zero solution is UAS and we give an estimate of the asymptotic stability region. Let $\varepsilon = \varepsilon_0$. Then we may choose suitable $\rho > 1$ such that

$$1 - \rho\varphi(H)(|A_1| + \varepsilon_0|B_1|) > 0.$$

Let

$$\begin{aligned}
\mu_0 &\equiv \frac{|A_0|}{1 - \rho\varphi(H)(|A_1| + \varepsilon_0|B_1|)}, \\
P_0 &\equiv \lambda_{\min}(C) - 2\mu_0\rho\varphi(H)|A_0^\top HA_1| - \mu_0^2\rho^2\varphi^2(H)\lambda_{\max}(H)|A_1|^2,
\end{aligned}$$

and

$$Q_0 \equiv 2\mu_0\rho\varphi(H)|B_1||HA_0| + 2\mu_0^2\rho^2\varphi^2(H)|B_1||HA_1| + \varepsilon_0\mu_0^2\rho^2\varphi^2(H)\lambda_{\max}(H)|B_1|^2.$$

It is easy to see that we may assume $\rho > 1$ is sufficiently close to 1 so that $\mu_0\rho > \mu$, and they are so close to each other that $P_0 > 0$.

Now choose an arbitrarily small number $\alpha > 0$ such that $0 < P_0 - \alpha < P$ and let

$$\delta_0 \equiv \frac{\min\{(P_0 - \alpha)/Q_0, \varepsilon_0\}}{\varphi(H)}. \quad (13)$$

Then for any solution $x(n) = x(n, n_0, \varphi)$ with $n_0 \in \mathbb{Z}^+$ and $\|\varphi\| < \delta_0$, by the same arguments as in Step I, we can derive that

$$\Delta V(x(n)) \leq -(P_0 - Q_0|x(n)|)|x(n)|^2, \quad (14)$$

whenever $\rho^2 V(x(n+1)) > V(x(s))$ for $n-r \leq s \leq n$. This implies

$$|x(s)| < \rho\varphi(H)|x(n+1)|, \quad \text{for } n-r \leq s \leq n.$$

We remark here that under the assumption $\rho^2 V(x(n+1)) > V(x(s))$ for $n-r \leq s \leq n$, it follows from (2) that $|x(n+1)| \leq \mu_0|x(n)|$.

On the other hand, as in Step I, we have that

$$\lambda_{\min}(H)|x(n)|^2 \leq V(x(n)) < \lambda_{\max}(H)\delta_0^2 \leq \lambda_{\max}(H)\frac{(P_0 - \alpha)^2}{Q_0^2\varphi^2(H)}, \quad \text{for all } n \geq n_0,$$

and thus,

$$|x(n)| < \frac{P_0 - \alpha}{Q_0}, \quad \text{for all } n \geq n_0.$$

Hence, (14) turns out to be

$$\Delta V(x(n)) \leq -\alpha|x(n)|^2, \quad \text{when } P(V(x(n+1))) > V(x(s)) \text{ for } n-r \leq s \leq n,$$

where $P(u) = \rho^2 u$ is as required in (ii) in Theorem 1.

Therefore, the zero solution of (2) is UAS by Theorem 1. Moreover, the asymptotic stability region contains at least the ball S_R with $R = \delta_0$ as given in (13). The proof is now complete.

It may happen that A_0 is not a stable matrix required by Theorem 2, but $A = A_0 + \theta A_1$ with some constant $\theta : 0 \leq \theta \leq 1$ is stable. Then we can still establish a similar theorem to Theorem 2.

For any given positively definite symmetric matrix C , there exists a positively definite symmetric matrix H such that

$$C = H - A^\top H A. \quad (15)$$

As in Theorem 2, we define

$$V(x) = x^\top H x,$$

then an extension of Theorem 2 can be obtained.

THEOREM 3. *Assume that A is stable and there holds the following two conditions:*

- (i) $1 - \varphi(H)|A_1| > 0$, where $\varphi(H) = \sqrt{\lambda_{\max}(H)/\lambda_{\min}(H)}$, and
- (ii) $P \equiv \lambda_{\min}(C) - 2(\bar{\mu}\varphi(H) + \theta)|A^\top H A_1| - (\bar{\mu}\varphi(H) + \theta)^2 \lambda_{\max}(H)|A_1|^2 > 0$, where

$$\bar{\mu} = \frac{|A_0| + |B_1|}{1 - \varphi(H)|A_1|}.$$

Then we have following three conclusions:

- (a) the zero solution of (2) is UAS for arbitrary $\tau > 0$;
- (b) any solution $x(n, n_0, \varphi)$ does not leave the ball $S_\varepsilon = \{x : |x(n, n_0, \varphi)| < \varepsilon\}$, for all $n \geq n_0$, whenever $\|\varphi\| < \delta(\varepsilon)$ with

$$\delta(\varepsilon) = \frac{\min\{(P/Q), \varepsilon\}}{\varphi(H)}, \quad (16)$$

where P as in (ii) and

$$Q \equiv 2\bar{\mu}\varphi(H)|B_1||HA| + 2\bar{\mu}\varphi(H)(\bar{\mu}\varphi(H) + \theta)|B_1||HA_1| + \bar{\mu}^2\varphi^2(H)\lambda_{\max}(H)|B_1|^2;$$

and

- (c) the asymptotic stability region Ω contains at least a ball S_R with the radius

$$R = \frac{\min\{(P_0 - \alpha)/Q_0, 1\}}{\varphi(H)}, \quad (17)$$

where P_0 , Q_0 , and α will be specified later.

The proof of this theorem is almost identical to that of Theorem 2, and it is hence, omitted.

4. EXAMPLE

In this section, we present four interesting example which demonstrate the usage of our stability results and stability/instability behavior of the considered systems. The first example is in three dimensions and gives an estimation of the sizes of the stability region and asymptotic stability region. The other three examples are in one dimension. Both the sizes of the stability regions and the numerical solutions are calculated, and the instability numerical results are particularly interesting. For one-dimensional equations, the stability problem becomes more transparent, and hence, our intuition can play a better role. In Example 2, we show the following two features in a system:

- (i) the initial disturbance vanishes as $n \rightarrow \infty$ (i.e., asymptotically stable), and
- (ii) the initial disturbance is amplified but bounded for all n (i.e., unstable but bounded).

In Example 3, a much longer delay is imposed. Besides the two features in Example 2, we show a third feature: the initial disturbance is amplified and eventually goes to infinity (i.e., unstable and unbounded). Example 4 shows a system with variable delays. All the above three evolution features of an initial disturbance are displayed.

EXAMPLE 1. Consider the following three-dimensional system of delay difference equations:

$$x(n+1) = A_0x(n) + A_1x(n - \tau_1(n)) + X(n - \tau_2(n))B_1x(n).$$

The three matrices A_0 , A_1 , and B_1 are

$$A_0 = \begin{bmatrix} 0.8 & 0.0 & 0.0 \\ 0.0 & 0.7 & 0.0 \\ 0.0 & 0.0 & -0.6 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.011 & 0.000 & 0.000 \\ 0.000 & 0.010 & 0.000 \\ 0.000 & 0.000 & -0.010 \end{bmatrix},$$

and

$$B_1 = \begin{bmatrix} 0.025 & -0.017 & 0.008 & 0.052 & -0.025 & 0.062 & 0.000 & -0.102 & 0.043 \\ -0.017 & -0.003 & 0.019 & -0.025 & -0.004 & 0.031 & -0.102 & 0.063 & -0.026 \\ 0.008 & 0.019 & 0.015 & 0.062 & 0.031 & -0.072 & 0.043 & -0.026 & -0.007 \end{bmatrix}^T,$$

respectively. Estimate the size of the stability region.

The first step in applying Theorem 2 is to solve the matrix equation

$$C = H - A_0^\top H A_0, \quad (19)$$

for H to any given positively definite symmetric matrix C .

The matrix C is chosen as

$$C = \begin{bmatrix} 3.0 & 1.0 & 1.2 \\ 1.0 & 2.0 & 0.4 \\ 1.2 & 0.4 & 1.0 \end{bmatrix}.$$

Then we can obtain from (19) that

$$H = \begin{bmatrix} 8.3333 & 2.2727 & 0.8108 \\ 2.2727 & 3.9216 & 0.2817 \\ 0.8108 & 0.2817 & 1.5625 \end{bmatrix},$$

and the eigenvalues of H are 9.3884, 2.9623, and 1.4667.

Thus,

$$\varphi(H) = \sqrt{\frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}} = 2.5301,$$

and by the definition of the spectral norm of matrix, we have

$$|A_0| = 0.8, \quad |A_1| = 0.011, \quad \text{and} \quad |B_1| = 0.0292.$$

Clearly, Condition (i) in Theorem 2, i.e., $1 - \varphi(H)|A_1| > 0$ is satisfied.

Now, if we let $\varepsilon_0 = 0.6849$, then we can calculate μ as follows:

$$G \equiv 1 - \varphi(H)[|A_1| + \varepsilon_0|B_1|] = 0.9126 \quad \text{and} \quad \mu = \frac{|A_0|}{G} = 0.8681.$$

For convenience, we denote

$$\gamma \equiv \mu\varphi(H)|A_1|,$$

and obviously, there holds

$$P > \lambda_{\min}(C) - 2|A_0^\top H|\gamma - \lambda_{\max}(H)|A_1|^2. \quad (20)$$

Now we compute the following:

$$\lambda_{\min}(C) = 0.4347, \quad |A_0^\top H| = 7.3615, \quad \gamma = 0.0242, \quad \lambda_{\max}(H) = 9.3884, \quad \text{and} \quad |A_1| = 0.011.$$

Substituting these into the right-hand side of (20), we obtain

$$P > 0.0717 > 0.$$

Hence, Condition (ii) in Theorem 2 is also satisfied. (Note that here $\varepsilon_0 = 0.6849 < 1$ and $G = 1 - \varphi(H)[|A_1| + \varepsilon_0|B_1|] = 0.9126 > 0$.)

Therefore, the zero solution of system (18) is UAS. In addition, upon obtaining Q , P_0 , and Q_0 , we can derive that

$$\delta(\varepsilon) = \frac{\varepsilon}{\varphi(H)} = 0.3952\varepsilon \quad \text{and} \quad R = \frac{\varepsilon_0}{\varphi(h)} = 0.2707.$$

Here, $\delta(\varepsilon)$ and R correspond to the Conclusions (b) and (c) of Theorem 2, respectively, where $\delta(\varepsilon)$ gives an estimate of the size of the stability region. Namely, as long as the norm of the initial disturbance is less than $\delta = 0.3952\varepsilon$, the norm of the solution is always less than ε . An estimate

of the asymptotic stability region is given by $R = 0.2707$. As long as the norm of the initial disturbance is less than 0.2707, the norm of the solution approaches zero as $n \rightarrow \infty$.

EXAMPLE 2. Consider the scalar delay difference equation of the form

$$x(n+1) = -0.60x(n) + a_1x(n-2) + 0.15x(n-3)x(n), \quad n \geq 0, \quad (21)$$

and the associated initial conditions

$$x(-3) = 0.8, \quad x(-2) = 0.2, \quad x(-1) = -0.4, \quad x(0) = -0.7. \quad (22)$$

Now $k = 1$. We leave a_1 as the tuning parameter for stability studies and denote

$$A_0 = a_0 = -0.60, \quad A_1 = a_1, \quad B_1 = b_1 = 0.15, \quad H = h, \quad \text{and} \quad C = c.$$

Then (2) turns out to be

$$c = h(1 - a_0^2),$$

which implies

$$h = \frac{c}{1 - a_0^2}.$$

For convenience, choose $c = 1$ to obtain $h = 1.5625$. We discuss two cases as follows.

THE STABLE CASE. Let $a_1 = 0.1$ and $\varepsilon_0 = 0.9$. Then, it can be verified that

- (i) $1 - \varphi(H)|A_1| = 1 - |a_1| = 0.9 > 0$;
- (ii) $G = 1 - \varphi(H)[|A_1| + \varepsilon_0|B_1|] = 1 - [|a_1| + \varepsilon_0|b_1|] = 0.765 > 0$,

$$\mu = \frac{|a_0|}{G} = 0.7843,$$

$$\begin{aligned} P &= \lambda_{\min}(C) - 2\mu\varphi(H)|A_0^\top H A_1| - \mu^2\varphi^2(H)\lambda_{\max}(H)|A_1|^2 \\ &= |c| - 2\mu|a_0|ha_1 - \mu^2|h||a_1|^2 = 0.8433 > 0, \end{aligned}$$

and

$$\begin{aligned} Q &= 2\mu\varphi(H)|B_1||H A_0| + 2\mu^2\varphi^2(H)|B_1||H A_1| + \mu^2\varphi^2(H)\lambda_{\max}(H)|B_1|^2 \\ &= 2\mu|b_1||ha_0| + 2\mu^2|b_1||ha_1| + \mu^2|h||b_1|^2 = 0.2704. \end{aligned}$$

Hence, by applying Theorem 2, we can conclude that the zero solution of (21) is UAS, and

$$\delta(\varepsilon) = \frac{\min\{(P/Q), \varepsilon\}}{\varphi(H)} = \min\left\{\frac{0.8433}{0.2704}, \varepsilon\right\} = \min\{3.1125, \varepsilon\} = \varepsilon, \quad \text{with } \varepsilon \leq \varepsilon_0 < 1,$$

while the asymptotic stability region Ω contains at least a ball S_{ε_0} with $\varepsilon_0 = 0.9$.

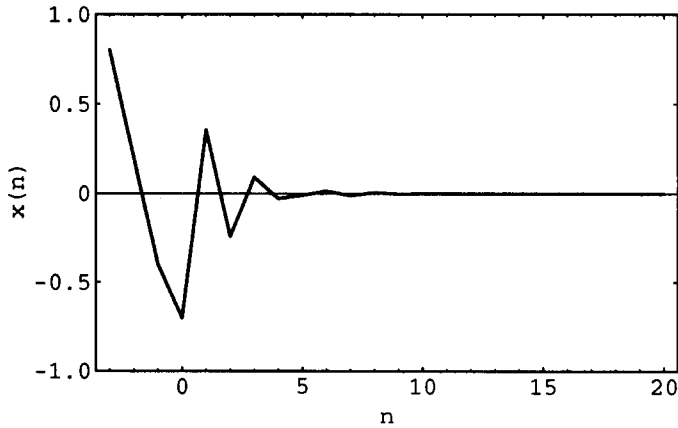


Figure 1. The numerical solution of (21) with initial condition (22), where $a_1 = 0.1$. This solution implies that the zero solution of (21) is stable since the initial disturbance decays to zero.

Using *Mathematica*, the unique numerical solution of the initial value problem of this example is found. The solution is depicted in Figure 1, which shows that the solution approaches zero as $n \rightarrow \infty$. This agrees with the conclusion of Theorem 2.

THE UNSTABLE BUT BOUNDED CASE. We increase a_1 from 0.1 to 0.9 and choose $\varepsilon_0 = 0.1$. Then

- (i) $1 - \varphi(H)|A_1| = 0.1 > 0$;
- (ii) $G = 0.085 > 0$, $\mu = 7.0588$, but $P = -73.9736 < 0$.

Hence, Condition (ii) in Theorem 2 is not satisfied, and we are not able to assert that the zero solution of (21) is UAS.

The numerical solution of (21) is found for the following initial conditions:

$$x(-3) = 0.05, \quad x(-2) = 0.07, \quad x(-1) = -0.04, \quad x(0) = -0.06,$$

as shown in Figure 2. This solution does not go to zero as $n \rightarrow \infty$ and the zero solution of (21) is unstable.

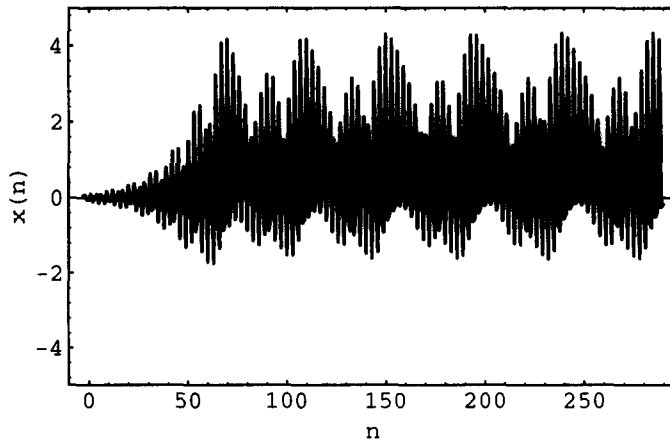


Figure 2. The numerical solution of (21) with initial condition (22), where $a_1 = 0.9$. This solution is bounded but does not go to zero, and hence, it implies that the zero solution of (21) is unstable since the initial disturbance does not decay to zero.

This example reveals an important fact that even though Conditions (i) and (ii) in Theorem 2 are sufficient, but not necessary for the zero solution being UAS, the obtained stability conditions are of reasonable accuracy. In other words, if these conditions are not satisfied, then the zero solution is very likely unstable at least for the scalar case ($k = 1$).

It is worth to notice that h is proportional to c , so are P and Q . Hence, our conclusions are independent of the choice of the value of $c > 0$.

EXAMPLE 3. Consider the scalar difference equation with larger delay as follows:

$$x(n + 1) = -0.6x(n) + a_1x(n - 10) + 0.15x(n - 3)x(n), \quad n \geq 0. \quad (23)$$

Again, we choose $c = 1$ and then $h = 1.5625$. Consider two cases.

THE STABLE CASE. Let $a_1 = 0.4$ and $\varepsilon_0 = 0.9$. Then,

$$G = 0.465 > 0, \quad \mu = 0.8602, \quad \text{and} \quad P = 0.1699 > 0.$$

Hence, all the conditions in Theorem 2 are satisfied, and thus, the zero solution of (23) is UAS.

Let us examine the numerical solution now. Choose a set of initial conditions, say, as follows:

$$\begin{aligned} x(-10) &= 0.8, & x(-9) &= 0.2, & x(-8) &= -0.4, & x(-7) &= -0.7, \\ x(-6) &= -0.3, & x(-5) &= 0, & x(-4) &= -0.8, & x(-3) &= -0.3, \\ x(-2) &= 0.1, & x(-1) &= -0.1, & x(0) &= 0.8. \end{aligned} \quad (24)$$

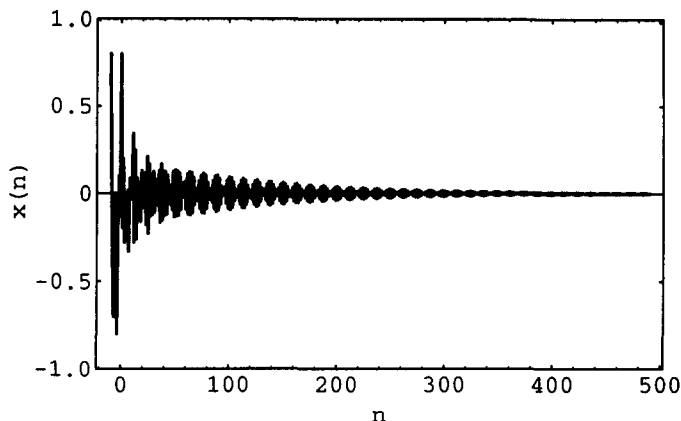


Figure 3. The numerical solution of (23) with initial condition (24), where $a_1 = 0.4$.

The graph of the corresponding numerical solution is shown in Figure 3. It indicates that the solution approaches zero as $n \rightarrow \infty$ and agrees with the conclusion of Theorem 2.

THE UNSTABLE BUT BOUNDED CASE. Let $a_1 = 0.5$ and $\varepsilon_0 = 0.1$. Then,

$$G = 0.485 > 0, \quad \mu = 1.0309, \quad \text{and} \quad P = -0.3816 < 0.$$

Hence, Condition (ii) in Theorem 2 is not satisfied.

Moreover, we note that no matter how small the chosen $\varepsilon_0 > 0$ is, even if $\varepsilon_0 = 0$, we still obtain $P < 0$.

For numerical solution, we choose the initial conditions

$$\begin{aligned} x(-10) &= 0.08, & x(-9) &= 0.02, & x(-8) &= -0.04, & x(-7) &= -0.07, \\ x(-6) &= -0.03, & x(-5) &= 0, & x(-4) &= -0.08, & x(-3) &= -0.03, \\ x(-2) &= 0.01, & x(-1) &= -0.01, & x(0) &= 0.08. \end{aligned} \quad (25)$$

The corresponding solution is shown in Figure 4, which demonstrates that it is unstable but it does not seem to become unbounded.

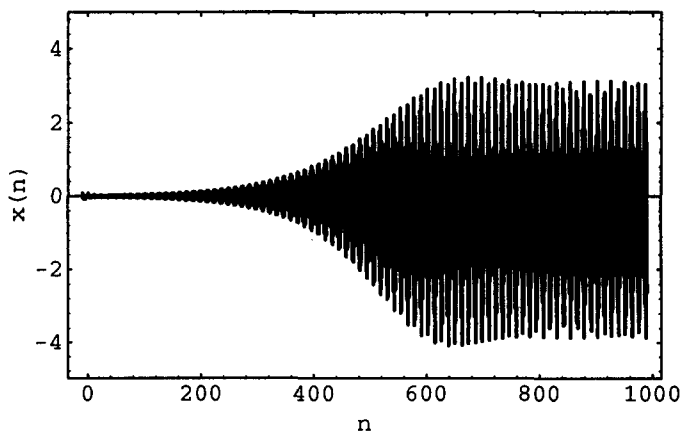


Figure 4. The same as Figure 3, except $a_1 = 0.5$.

THE UNSTABLE BUT BOUNDED CASE. Let $a_1 = 0.6$ and $\varepsilon_0 = 0.1$. Then,

$$G = 0.385 > 0, \quad \mu = 1.5584, \quad \text{and} \quad P = -2.1193 < 0.$$

Again, no matter how small $\varepsilon_0 > 0$ is (even if $\varepsilon_0 = 0$), we still have $P < 0$.

Hence, Condition (ii) in Theorem 2 is not satisfied. Now the numerical solution of the delay difference equation with the initial conditions (25) shows that the solution is unstable and unbounded. See Figure 5.

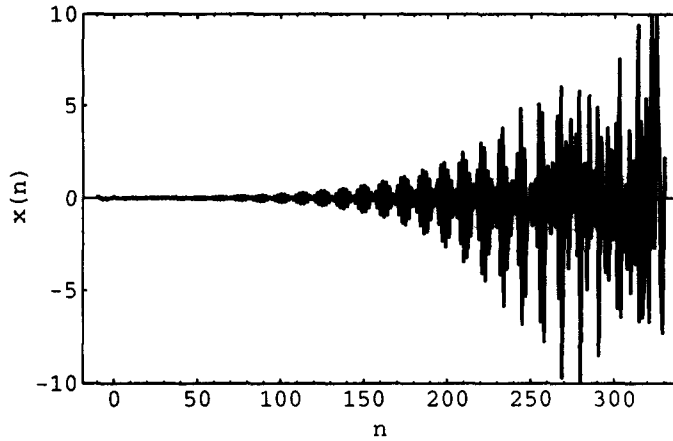


Figure 5. The same as Figure 3, except $a_1 = 0.6$.

EXAMPLE 4. Finally, we consider a scalar difference equation with variable delays as follows:

$$x(n + 1) = -0.6x(n) + a_1x(n - \tau_1(n)) + 0.15x(n - \tau_2(n))x(n), \quad n \geq 0, \quad (26)$$

where

$$\tau_1(n) = 1 + [\sin n] \quad \text{and} \quad \tau_2(n) = 2 + (-1)^n, \quad n \geq 0,$$

with $[\cdot]$ denoting the greatest integer function.

Note that $0 \leq \tau_i(n) \leq 3$, for $n \geq 0$ and $i = 1, 2$. Clearly, $a_1 > 1$ violates our Condition (i): $1 - \varphi(H)|A_1| = 1 - |a_1| > 0$. Thus, the conclusions of Theorem 2 do not apply. However, as mentioned above, the conditions of Theorem 2 are only the sufficient but not the necessary conditions for the zero solution to be UAS. Therefore, we may still have stable zero solutions when the conditions of Theorem 2 are not satisfied. Three cases are presented below.

THE STABLE CASE. $a_1 = 1.6$ and $\varepsilon_0 = 0.9$. The numerical solution of (26) with the initial conditions (22) shows that the solution tends to zero as $n \rightarrow \infty$. See Figure 6.

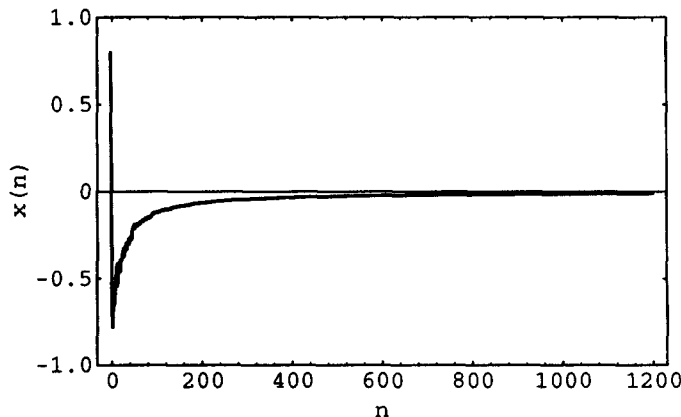


Figure 6. The numerical solution of (26) with initial condition (22), where $a_1 = 1.6$.

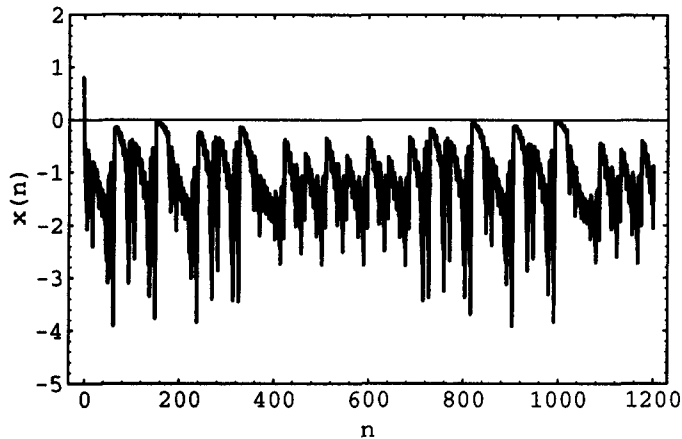


Figure 7. The same as Figure 6, except $a_1 = 1.915$.

THE STABLE CASE. $a_1 = 1.915$ and $\varepsilon_0 = 0.9$. The numerical solution of (26) with the same initial conditions (22) shows that the solution does not tend to zero but remains bounded as $n \rightarrow \infty$. See Figure 7.

THE UNSTABLE BUT BOUNDED CASE. We now increase a_1 by a very small number 0.002 from 1.915 to 1.917 and assign $\varepsilon_0 = 0.9$. The corresponding solution with the same initial conditions as used above is unbounded. See Figure 8.

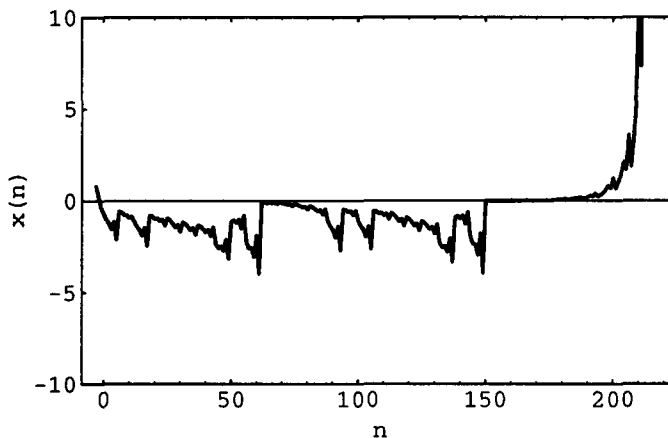


Figure 8. The same as Figure 6, except $a_1 = 1.917$.

It is noted that in the variable delay case, the stability of the zero solution of the equation is very sensitive to the coefficient a_1 of the linear term with delay. With only a little bit change in a_1 , while keeping all the other parameters unchanged, the behavior of the solution of (26) may vary drastically.

5. CONCLUSIONS AND DISCUSSIONS

We have carried out a quantitative study on the stability of zero solutions of the quadratic delay difference systems. Our goal was to estimate the sizes of the stability region and asymptotic stability region of the zero solution. When the initial disturbance is in the asymptotic stability region, the corresponding solution of the initial value problem of the quadratic delay difference system approaches zero. Examples are given for a three-dimensional system and three one-dimensional equations. Numerical results of these examples confirm the conclusions of the main theorem proved in this paper. Examples 2–4 show that when parameters in the systems are chosen so that the zero solution is unstable, when the stability conditions are not satisfied.

A curious question is: in an asymptotically stable case if the initial disturbance falls outside of the stability region, would the solution of the corresponding initial value problem still go to zero? This question was investigated numerically. We considered (23) in Example 3 with $a_1 = 0.5$. Hence, it is a stable case. But now choose the initial data far outside of the δ ball:

$$\begin{aligned} x(-10) &= 7.8, & x(-9) &= 4.2, & x(-8) &= -7.4, & x(-7) &= -3.7, \\ x(-6) &= -9.3, & x(-5) &= 7.0, & x(-4) &= -5.8, & x(-3) &= -9.3, \\ x(-2) &= 16.6, & x(-1) &= -9.1, & x(0) &= 3.8. \end{aligned} \tag{27}$$

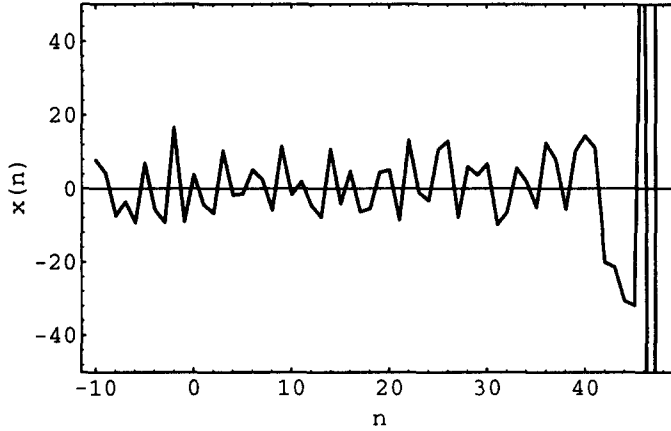


Figure 9. The numerical solution of (23) with large initial data (27), where $a_1 = 0.4$. This solution demonstrates that the zero solution is not a global attractor.

The numerical solution is shown in Figure 9 and is unbounded. This example demonstrates that although the zero solution is stable, there are some large initial disturbances which can be amplified and become infinity (instead of going to zero) as $n \rightarrow \infty$. Thus, the zero solution of (23), although asymptotically stable, may not be a global attractor. We hence conclude that our quantitative study of the stability criteria and estimation of the size of the stability region are of reasonable accuracy.

Another question is the relative importance of the first delay term measured by the matrix A_1 and the nonlinear term measured by the matrix B_1 . In our numerical experiments for the above examples, when the disturbance is small, the coefficient A_1 of the first delay term plays a more important role than the nonlinear term. It remains an interesting question when the system's stability becomes more sensitive to B_1 than A_1 .

As for the technical details of this paper, we wish to make the following discussions. It is easy to see that for a given system of the form (2), one may choose an arbitrary positively definite symmetric matrix C to get the corresponding matrix H satisfying (3) or (16). Then under Conditions (i) and (ii) in Theorems 2 or 3, one can find the corresponding number $\delta(\varepsilon)$ in the US for each sufficiently small number $\varepsilon > 0$; moreover, one can calculate the radius R of the ball inscribed in the asymptotic stability region. It is clear that for different choices of C , the corresponding values of $\delta(\varepsilon)$ and R are different. Hence, it remains to be solved that what is the best possible choice of C so that the $\delta(\varepsilon)$ or R attains the maximum.

The values P and Q in Theorems 2 or 3 are independent of ε . Hence, for any given $\varepsilon > 0$ with $\varepsilon \leq P/Q$, it follows from (4) that

$$\delta(\varepsilon) = \frac{\varepsilon}{\varphi(H)},$$

which gives us the immediate relationship between ε and $\delta(\varepsilon)$.

The required parameter θ in Theorem 3 is not unique. It should be regarded as one of the distinct advantages of our results. Because we have much freedom to suitably choose θ to obtain

better estimates for the size of the asymptotic stability regions. If the matrix A_0 itself is stable, then one may apply Theorem 2; while if A_0 is not stable, then one may try to choose a θ value so that $A = A_0 + \theta A_1$ is stable and then apply Theorem 3. Even if A_0 is stable and Theorem 2 applies, one can still use Theorem 3 with a proper choice of θ . Then the question is what is the best choice of θ to optimally estimate the asymptotic stability regions, or whether such a choice of θ will help an optimal estimation of the asymptotic stability regions at all.

The results obtained in this work are independent of the length of the delay $\tau > 0$. Hence, our results are a kind of unconditional stability for quadratic delay difference systems.

The arguments used in this work can be extended to the following quadratic delay difference systems of the general form:

$$x(n+1) = A_0x(n) + A_1x(n - \tau_1(n)) + X(n)B_0x(n) \\ + X(n - \tau_2(n))B_1x(n) + X(n - \tau_3(n))B_2x(n - \tau_4(n)),$$

where $n \in \mathbb{Z}^+$, $x \in \mathbb{R}^k$, A_0 , A_1 , $X(n)$ are the same as in (2), B_j^\top ($j = 0, 1, 2$) are all $k \times k^2$ matrices with the same form as B_1 in (2), and $\tau_j : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ with $0 \leq \tau_j(n) \leq r$ for some positive integer r , ($j = 1, 2, 3, 4$). Numerically there is no essential difference in calculating the values of $\delta(\varepsilon)$ and R between the simple form of (2) and the general form as above.

Elaydi [6] also described certain quantitative results on stability but for autonomous delay differential equations. The authors used a geometric argument to demonstrate their results. Our current work benefited much from Elaydi [6] in terms of motivation, yet we have taken a different approach when demonstrating our results and our working objective is delay difference equations. Our proof is an analytic one which makes Conclusions (b) and (c) in Theorem 2 more transparent. In addition, the system (2) is not autonomous since the delays $\tau_i(n)$ may not be constants. Hence, the results obtained in this paper are more general than those in [6].

At the end, we may conclude that it is feasible to carry out meaningful quantitative studies on the stability of the delay difference systems. We feel that our method will have applications in the numerical methods for solving continuous dynamical systems, automata lattice, and other areas of natural sciences and engineering.

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