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Notes

On the limit of subcritical free-surface flow over an obstruction

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Summary. In this note, through studying an asymptotically reduced equation which is a forced Korteweg-de Vries equation, an analytical expression of Forbes' hydraulic fall solution is obtained. It is pointed out that Forbes' solution of a hydraulic fall over an obstruction is the limit of cnoidal wave solutions as the upstream Froude number approaches a certain value less than one from below.

1 Introduction

Recently, Forbes [1] used the boundary integral method to solve an exact problem of the Laplace equation with nonlinear boundary conditions, and found a hydraulic fall solution over a semi-circle obstruction for a subcritical upstream flow of a certain Froude number. The solution indicates that the Froude number changes from subcritical upstream to supercritical downstream. Hence, Forbes took the upstream Froude number as a part of the solution.

Forbes and Schwartz [2] computed subcritical flows for the same model and found that the free surface of the downstream flow consists of cnoidal waves. According to linear theory (Lamb [3]), the downstream free surface is made of sinusoidal waves if the upstream subcritical Froude number is less than one, and such a model does not have a steady state solution if the Froude number is equal to one. That cnoidal waves approach a solitary wave (sinusoidal waves) in the limit as the period goes to infinity (zero respectively), is wellknwon. However, the limit of the downstream cnoidal waves of Forbes and Schwartz is not known yet.

Based upon the above facts, one must ask what the limiting case of subcritical fluid flow over an obstruction as the Froude number increases is. It is this point that stimulates us to write this note. Our answer to this question, as the main result of this note, is: For a given obstruction, there exists a limit of the subcritical upstream Froude number F_L such that the downstream flow consists of cnoidal waves, if $-\infty < F < F_L$. For $F = F_L$, the downstream flow is wave free and supercritical. Hence a hydraulic fall over the obstruction occurs as $F = F_L$.

Therefore, Forbes' result [1] is the limit of the subcritical flows of Forbes and Schwartz [2]. In the following, we will justify this conclusion.

2 Derivation of the main results

Consider an ideal fluid flow passing an obstruction in a two-dimensional open channel (Fig. 1). The governing equations are as follows:

$$u^*_{x^*} + v^*_{y^*} = 0, \tag{1}$$

$$\varrho(u^* u^*_{x^*} + v^* u^*_{y^*}) = -p^*_{x^*}, \tag{2}$$

$$\varrho(u^* v^*_{x^*} + v^* v^*_{y^*}) = -p^*_{y^*} - \varrho g; \tag{3}$$

at the free surface $y^* = H + \eta^*$,

$$u^* \eta^*_{x^*} - v^* = 0, \qquad p^* = 0; \tag{4}$$

at the bottom $y^* = h^*(x^*)$,

$$u^* h^*_{x^*} - v^* = 0, \tag{5}$$

where (u^*, v^*) is the velocity, p^* is the pressure, ϱ is the constant density, g is the constant gravitational acceleration. We introduce the following nondimensional variables:

$$(u, v) = (u^*, \varepsilon^{-1/2} v^*)/\sqrt{gH}, \qquad (x, y) = (\varepsilon^{1/2} x^*, y^*)/H,$$

$$p = p^*/(\varrho g H), \qquad \eta = \eta^*/H, \qquad h = \varepsilon^{-2} h^*/H,$$

$$\varepsilon = (H/L)^2 \ll 1 .$$

Here H and L are taken as vertical and horizontal scales respectively. Assume that u, v, p and η possess asymptotic expansions of the form

$$\varphi = \varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \cdots \tag{6}$$

with $u_0 = 1$, $v_0 = 0$, $p_0 = 1 - y$, $\eta_0 = 0$. Let the upstream Froude number $F \left(= \dfrac{C}{\sqrt{gH}} \right)$ be

$$F = 1 + \varepsilon F_1, \qquad F_1 < 0 \text{ (subcritical)}. \tag{7}$$

Then one can readily derive an equation for η_1 which is a forced stationary Korteweg-de Vries equation as follows [4]:

$$F_1 \eta_{1x} - \frac{3}{2} \eta_1 \eta_{1x} - \frac{1}{6} \eta_{1xxx} = h_x/2, \qquad x > x_- \tag{8}$$

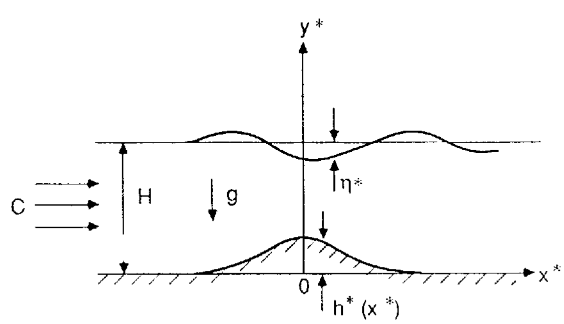

Fig. 1. Configuration of the fluid domain

with

$$\eta_1(x) = \eta_1'(x) = 0 \qquad \text{as} \quad x \leqq x_-, \tag{9}$$

where h is assumed to be of compact support with supp $(h) = [x_-, x_+]$.

Define a complete metric space

$$B = \{f \mid f \in C([x_-, x_+]), \quad \|f\| = \max_{x_- \leq x \leq x_+} |f(x)| \leq M \quad \text{for some given positive constant } M\}.$$

For $x_- \leqq x \leqq x_+$, Eqs. (8) and (9) can be converted into an integral equation

$$\eta_1(x) = -\frac{1}{\sqrt{-6F_1}} \int_{x_-}^{x} \sin \sqrt{-6F_1}\,(x-\tau) \left[\frac{9}{2}\,\eta_1^2(\tau) + 3h(\tau)\right] d\tau. \tag{10}$$

By using the contraction map theorem in B, we can easily show that if F_1 satisfies

$$\left(3/\left(2\sqrt{-6F_1}\right)\right)(3M + 6\,\|h\|/M)\,(x_+ - x_-) \leqq 1 \tag{11}$$

and

$$\left(9M/\sqrt{-6F_1}\right)(x_+ - x_-) < 1, \tag{12}$$

then (10) has a solution in B. For given h and M, Eqs. (11) and (12) can always be satisfied as long as $|F_1|$ is sufficiently large.

Having established existence of a solution to Eqs. (8) and (9) from x_- to x_+, we turn to find the extension of the solution to $[x_+, \infty)$. This is equivalent to solving the following initial value problem:

$$\frac{1}{3}\,(\eta_{1x})^2 = P(\eta_1) - D, \qquad x > x_+, \tag{13}$$

$$\eta_1(x_+) = \eta_1^+ \tag{14}$$

where

$$P(\eta_1) = \eta_1^2(2F_1 - \eta_1), \tag{15}$$

$$D = D(F_1, h) = \frac{1}{3}\,[-(\eta_{1x}^+)^2 + 6F_1(\eta_1^+)^2 - 3(\eta_1^+)^3], \tag{16}$$

and η_1^+ and η_{1x}^+ are obtained from the solution of (10) at $x = x_+$.

It is known that the above initial value problem has a cnoidal wave solution, a wave free solution or an unbounded solution depending upon $P(\eta_1) = D(F_1, h)$ having three distinct roots, a double root, or only one real root [5]. One can readily show that $P(\eta_1) = D$ has a double root for $D \neq 0$ if and only if $F_1 = F_L \leqq 0$ where

$$D(F_L, h) = \frac{32}{27}\,F_L^3. \tag{17}$$

Numerically we found that $0 > D(F_1, h) > \frac{32}{27}\,F_1^3$ if $F_1 < F_L$ (cnoidal wave solution), and $D(F_1, h) > 0$ if $0 \geqq F_1 > F_L$ (unbounded solution), for

$$h(x) = \begin{cases} R\sqrt{1 - x^2}, & |x| \leqq 1 \\ 0, & |x| > 1 \end{cases} \tag{18}$$

with $R > 0$. It is condition (17) that determines the hydraulic fall free surface of Forbes. For a given R, we can find an F_L. Numerical results are shown in Fig. 2.

For $F_1 < F_L$, the solution to Eqs. (13) and (14) can be expressed as

$$\eta_1(x) = -\frac{4F_1}{3}\left[\cos\left(\theta + \frac{4\pi}{3}\right) - \frac{1}{2} + \left(\cos\theta - \cos\left(\theta - \frac{4\pi}{3}\right)\right)\right.$$
$$\left.\times\, cn^2\left(\sqrt{F_1\left(\cos\left(\theta + \frac{\pi}{3}\right) - \cos\theta\right)}\,(x - x_0)\right)\right], \qquad x > x_+, \tag{19}$$

where

$$\theta = \frac{1}{3}\arccos\left[27D/(16F_1^3) - 1\right], \qquad 0 \leq \theta \leq \frac{\pi}{3}. \tag{20}$$

The phase shift x_0 will be determined by (24). The period of the above cnoidal wave is

$$T = 2\left[\sqrt{F_1\left(\cos\left(\theta + \frac{2\pi}{3}\right) - \cos\theta\right)}\right]^{-1} K(k^2) \tag{21}$$

with

$$k^2 = \frac{\cos\theta - \cos\left(\theta + \frac{4\pi}{3}\right)}{\cos\theta - \cos\left(\theta + \frac{2\pi}{3}\right)} < 1, \tag{22}$$

and

$$K(k^2) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2\sin^2 t}}. \tag{23}$$

In order to make (19) satisfy (14), the phase shift x_0 must satisfy

$$\eta_1^+ = -\frac{4F_1}{3}\left[\cos\left(\theta + \frac{4\pi}{3}\right) - \frac{1}{2}\right.$$
$$\left. + \left(\cos\theta - \cos\left(\theta + \frac{4\pi}{3}\right)\right) cn^2\left(\sqrt{F_1\left(\cos\left(\theta + \frac{2\pi}{3}\right) - \cos\theta\right)}\,(x_+ - x_0)\right)\right]. \tag{24}$$

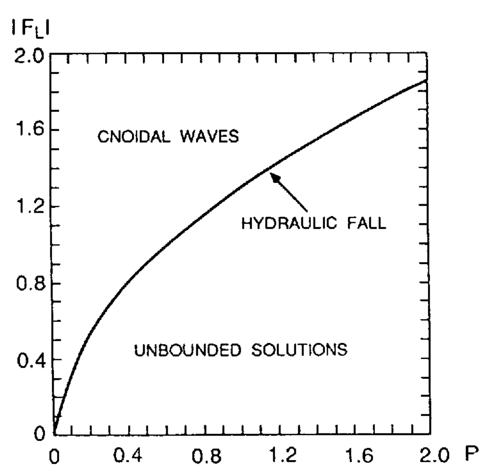

Fig. 2. Solution curve of the condition (17) where R characterizes the obstruction shape h by the equation (18)

There are two limiting cases:

(i) As $F_1 \to -\infty$, $D/F_1^3 \approx 0$, $\theta \approx \pi/3$, $k^2 \approx 0$, and $K(k^2) \approx K(0) = \pi/2$. By (21), the period T is $2\pi/\sqrt{-6F_1}$ which corresponds to sinusoidal wave.

(ii) As (17) holds, $\theta = 0$, $k^2 = 1$, and $K(k^2) = K(1) = \infty$. By (21), the period is $T = \infty$, which corresponds to the wave free solution of a hydraulic fall. This hydraulic fall can be expressed by

$$\eta_1(x) = -(4F_L/3)\left(-1 + \frac{3}{2}\,\mathrm{sech}^2 \sqrt{-3F_L/2}\,(x - x_0)\right). \tag{25}$$

This is the analytical expression of Forbes' hydraulic fall solution mentioned in the Summary.

By the way, we point out that a similar limit was analytically discovered by J. W. Miles [6] for a case of very short obstruction (see [6, Eqs. 4.6a, b]). Mathematically, the assumption of short obstruction in [6] is equivalent to $h(x) \propto \delta(x)$ in our case (see Eq. (8) of this paper), where $\delta(x)$ is the Dirac delta function. Hence the limit found in [6] may be considered as a special case of our present paper.

Three typical free surface profiles for $R = 1.0$ and $F_1 = -4.0$ (sinusoidal wave), $F_1 = -1.4$ (cnoidal wave), and $F_1 = -1.291561 = F_L$ (hydraulic fall) are shown in Fig. 3.

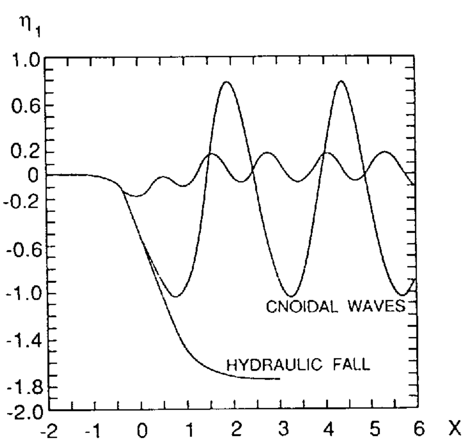

Fig. 3. Three typical solution profiles of Eqs. (8) and (9) with $h(x)$ defined by (18) for $R = 1.0$. The solution of the smallest amplitude corresponds to $F_1 = -4.0$. This solution is a approximately sinusoidal wave whose period can be computed from the formula $2\pi/\sqrt{-6F_1}$. The periodic solution of the larger amplitude is a cnoidal wave solution corresponding to $F_1 = -1.4$. The hydraulic fall solution corresponds to the limit of $F_1 = F_L = -1.291561$

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