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Estimation of the Global Mean Temperature with Point Gauges

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ABSTRACT

This paper considers the mean squared error (MSE) incurred in estimating an idealized earth's global average temperature with a finite number of point gauges located in a specified or stochastic way over the globe. For a class of model earths with rotationally invariant statistics, the MSE formula can be cast into the form of a summation over spherical harmonic indices. The summand factors into a part which depends on the design of the gauge network and a second part which is the degree variance spectrum of the surface temperature field. After presenting this formalism, we provide an example spectrum for the surface temperature field derived from a simple two parameter stochastic climate model defined on the sphere. An example calculation is given for the case of N gauges randomly arranged on the sphere. In addition, the sampling error is computed for some simple regular arrays of gauges as illustrative examples.

KEY WORDS: Global mean temperature; surface temperature field; spherical harmonics.

1. INTRODUCTION

Several groups around the world are attempting to estimate the trend in the apparently increasing global average temperature (*cf.* Hansen and Lebedeff 1987; Jones *et al.* 1986a and 1986b; Houghton *et al.* 1990), since it is

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now recognized that the increasing concentrations of certain trace gases have the potential to drive the temperature to even higher levels (Houghton *et al.* 1990). There are three main thrusts of physical science research concerning the so-called greenhouse effect: Estimation of the trends in the earth's climate based upon observations over the last century; Estimation of the trends in the greenhouse gases according to past observational data and future levels based upon assumed greenhouse gas emission scenarios; Formulation and study of climate models which simulate and forecast the trends.

In this paper we focus on a very small portion of the first of these problems: for an idealized fluctuation field on the sphere with a given network of isolated unbiased point gauges, what is the expected mean squared error in the estimation of the global average value of the temperature field? The problem lies in the fact that the gauges are separated by finite spatial distances and that these spatial gaps lead to an inevitable 'sampling error'. Complicating the problem is the fact that the field has correlations from one point on the sphere to another and that these correlations depend on the length of temporal averaging employed. We do not intend to present the final analysis of this problem here, but rather explore some simplified models of the procedure, both the field and the measurement design, in order to better understand the estimation issue.

Our attention is focused on an idealized planet whose surface temperature fluctuation statistics are rotationally invariant on the sphere (the analog of stationarity in time series analysis); we also discuss the limits in which such an analog is faithful. For a given temporal smoothing filter, only one function is necessary to specify such a field: the spatial autocorrelation function or equivalently, the spherical harmonic variance spectrum. We will formulate the mean squared error (MSE) for the problem as a sum over the spherical harmonic indices of a factor dependent only on the sampling design and another factor which is the spherical harmonic variance spectrum. The technique is the spherical version of a planar one employed in the analysis of area average rain rate estimation designs by North and Nakamoto (1989). A simple stochastic model of the temperature field will be developed which is useful as a guide in the problem. Finally, a few examples will be presented which give a feeling for the magnitude of error to be expected in some typical designs. The problem will then be cast into the perspective of taking the earth's average surface temperature within a specified tolerance.

2. THE GLOBAL AVERAGE TEMPERATURE

Let the temperature be designated by $\Theta(\hat{n}, t)$ at the point $\hat{n}$, a unit vector pointing from the sphere's center to the point in question, and at time t. We assume that the random temperature field is homogeneous and stationary. This assumption amounts to

$$\langle \Theta(\hat{\boldsymbol{n}}, t), \Theta(\hat{\boldsymbol{n}}', t') \rangle = \sigma_{ip}^2 \rho(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}', \tau) \tag{1}$$

where $\langle\rangle$ denotes the ensemble average, $\sigma_{ip}^2 = \langle \Theta^2(\hat{\boldsymbol{n}}, t) \rangle$ is the instantaneous point variance of the temperature field $\Theta(\hat{\boldsymbol{n}}, t)$, and $\tau = t - t'$ is the time lag. The space-time autocorrelation, $\rho(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}', \tau)$ is an even function of each of τ and obviously invariant under interchange of $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$.

Since in the present problem we are not especially interested in the time dependence, but rather the errors incurred because of the sparseness of the gauges, we will examine only the case of long term averages of the measurements (i.e., the low frequency limit). To define what is meant by this, it is convenient to use the (complex valued) Fourier transformation $\Theta(\hat{\boldsymbol{n}}, f)$ defined by

$$\widetilde{\Theta}(\hat{\boldsymbol{n}}, f) = \int_{-\infty}^{\infty} dt\, \Theta(\hat{\boldsymbol{n}}, t) e^{i2\pi f t} \quad . \tag{2}$$

In what follows we will generally be interested in the low frequency limit of the Fourier Transform, $\widetilde{\Theta}(\hat{\boldsymbol{n}}, 0)$, which is real valued. From an operational point of view we can find this quantity by passing the data through a moving average filter whose span is much longer than the autocorrelation time of the system (a decade or so). The dependence on the width of the averaging window is important to climatologists and will be dealt with in a separate paper. We assume that the process is well behaved and, without loss of generality, we assume a zero mean process. By "well behaved", we mean that the first two moments are finite and that the process satisfies smoothness assumptions such that a Karhünen-Loève expansion in spherical harmonics results in

$$\left\| \widetilde{\Theta}(\hat{\boldsymbol{n}}) - \sum_{n=0}^{L} \sum_{m=-n}^{n} \widetilde{\Theta}_n^m Y_n^m(\hat{\boldsymbol{n}}) \right\|^2 \xrightarrow{w.p.1} 0 \quad \text{as} \quad L \to \infty \tag{3}$$

where $\xrightarrow{w.p.1}$ means converges with probability one.

The global average of $\widetilde{\Theta}(\hat{\boldsymbol{n}})$ over a unit sphere is given by

$$\bar{\widetilde{\Theta}} = \frac{1}{4\pi} \int_{4\pi} d\Omega\, \widetilde{\Theta}(\hat{\boldsymbol{n}}) \quad . \tag{4}$$

Note that we have dropped the f dependence since we are henceforth dealing only with the $f \to 0$ limit.

Because of the spherical symmetry prescribed in this idealized problem, it is then convenient to expand the low frequency temperature field into spherical harmonics $Y_n^m(\hat{\boldsymbol{n}})$ (Arfken 1985), defined by

$$Y_n^m(\hat{\boldsymbol{n}}) \equiv \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}}\, P_n^m(\cos\theta) e^{im\phi} \quad , \qquad -n \le m \le n \tag{5}$$

where P_n^m are the standard associated Legendre functions and the normalization coefficients are chosen so that the $Y_n^m(\hat{\boldsymbol{n}})$ are orthonormal, i.e.,

$$\int_{4\pi} d\Omega\, Y_n^{m*}(\hat{\boldsymbol{n}}) Y_{n'}^{m'}(\hat{\boldsymbol{n}}) = \delta_{mm'}\delta_{nn'} \quad , \tag{6}$$

and we may expand $\widetilde{\Theta}(\hat{\boldsymbol{n}})$

$$\widetilde{\Theta}(\hat{\boldsymbol{n}}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \widetilde{\Theta}_n^m Y_n^m(\hat{\boldsymbol{n}}) \tag{7}$$

where

$$\widetilde{\Theta}_n^m = \int_{4\pi} d\Omega\, Y_n^{m*}(\hat{\boldsymbol{n}}) \widetilde{\Theta}(\hat{\boldsymbol{n}}) \quad . \tag{8}$$

The field $\widetilde{\Theta}(\hat{\boldsymbol{n}})$ can be modelled with realizations of a surface temperature field computed from a stochastic model. Later we will provide a simple model for the random field which can be used in explicit calculations.

The variables $\widetilde{\Theta}_n^m$ are complex valued random variables. In the special case that the second moment statistics of $\widetilde{\Theta}(\hat{\boldsymbol{n}})$ are rotationally invariant on the sphere,

$$\langle \widetilde{\Theta}(\hat{\boldsymbol{n}}) \widetilde{\Theta}(\hat{\boldsymbol{n}}') \rangle = \sigma^2 \rho(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}') \quad . \tag{9}$$

Here,

$$\sigma^2 = \langle \widetilde{\Theta}^2(\hat{\boldsymbol{n}}) \rangle \tag{10}$$

is the low frequency point variance of the field and is a constant due to the assumption that the field is stationary and statistically rotationally invariant.

The spatial autocorrelation function $\rho(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}')$ can be expanded into Legendre polynomials $P_n(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}')$ as

$$\rho(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}') = \sum_{n=0}^{\infty} (2n+1) \rho_n P_n(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}') \quad . \tag{11}$$

The addition theorem for spherical harmonics (Arfken 1985),

$$P_n(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}') = \frac{4\pi}{2n+1} \sum_{m=-n}^{n} Y_n^{m*}(\hat{\boldsymbol{n}}) Y_n^m(\hat{\boldsymbol{n}}') \quad , \tag{12}$$

can be used with (8) to obtain

$$4\pi\sigma^2 \rho_n \delta_{nn'} \delta_{mm'} = \langle \widetilde{\Theta}_n^{m*} \widetilde{\Theta}_{n'}^{m'} \rangle \quad . \tag{13}$$

3. ESTIMATOR OF THE GLOBAL TEMPERATURE AND SAMPLING ERRORS

Let $\hat{\boldsymbol{n}}_i$ denote the position of the i-th point gauge on the surface of the sphere, $i = 1, 2, \ldots, N$. Here, N is the total number of gauges. The true average of the random temperature field $\widetilde{\Theta}(\hat{\boldsymbol{n}})$ on the surface of the entire sphere is Ψ , and is obtained by

$$\Psi = \frac{1}{4\pi} \int_{4\pi} d\Omega \, \widetilde{\Theta}(\hat{\boldsymbol{n}}) \ . \tag{14}$$

In this formula, $d\Omega$ denotes the solid angle partition of the unit sphere and the integration for $d\Omega$ is over the surface of the entire unit sphere. The straight arithmetic average by these N gauges is employed as an unbiased estimator of the true global average temperature Ψ:

$$\begin{aligned} \Psi_N =& \frac{1}{N} \sum_{i=1}^{N} \widetilde{\Theta}(\hat{\boldsymbol{n}}_i) \\ =& \frac{1}{4\pi} \int_{4\pi} d\Omega \, \widetilde{\Theta}(\hat{\boldsymbol{n}}) K(\hat{\boldsymbol{n}}) \end{aligned} \tag{15}$$

where

$$K(\hat{\boldsymbol{n}}) = \frac{4\pi}{N} \sum_{i=1}^{N} \delta(\hat{\boldsymbol{n}} - \hat{\boldsymbol{n}}_i) \tag{16}$$

where the Dirac delta function notation is defined more conventionally as

$$\delta(\hat{\boldsymbol{n}} - \hat{\boldsymbol{n}}_i) \equiv \delta(\phi - \phi_i)\delta(\cos\theta - \cos\theta_i) \tag{17}$$

such that an area integral over the singularity gives unity.

The mean squared error (MSE) is then defined by the following ensemble average:

$$\epsilon^2 = \langle (\Psi - \Psi_N)^2 \rangle \ . \tag{18}$$

With the homogeneity assumption, $\langle \Theta(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}}') \rangle = \sigma^2 \rho(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}')$, this formula can be written as

$$\epsilon^2 = \frac{\sigma^2}{(4\pi)^2} \int d\Omega \int d\Omega' \, \rho(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}')(1 - K(\hat{\boldsymbol{n}}))(1 - K(\hat{\boldsymbol{n}}')) \ . \tag{19}$$

In this integral, the integration domain is $\Omega \times \Omega$ where Ω is the entire surface of a unit sphere. The expectation operator ($\langle \rangle$) may be moved under the integral in (18) by an application of Fubini's Theorem since all values are non-negative. Following equations (19) and (11), one can derive that

$$\epsilon^2 = \frac{\sigma^2}{(4\pi)^2} \sum_{l=0}^{\infty} (2l+1)\rho_l \int d\Omega \int d\Omega' \, P_l(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}')(1 - K(\hat{\boldsymbol{n}}))(1 - K(\hat{\boldsymbol{n}}')) \ . \tag{20}$$

Applying some familiar identities from the literature on spherical harmonics, e.g. (Arfken 1985), one can obtain a compact expression for the above double sphere integral:

$$\int d\Omega \int d\Omega' \, P_l(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}')(1 - K(\hat{\boldsymbol{n}}))(1 - K(\hat{\boldsymbol{n}}')) = (4\pi)^2 \left[\frac{1}{N^2} \sum_{i,j=1}^{N} P_l(\hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_j) - \delta_{l0} \right] \quad . \tag{21}$$

The mean square error may now be expressed:

$$\epsilon^2 = \sigma^2 \sum_{l=1}^{\infty} \frac{(2l+1)\rho_l}{N^2} \sum_{i,j=1}^{N} P_l(\hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_j) \quad . \tag{22}$$

Next, we examine the relationship between the low frequency point variance, σ^2, and the variance of the area average on the sphere, $\sigma_{\oplus}^2$. The quantity $\sigma_{\oplus}^2$ is defined as follows

$$\sigma_{\oplus}^2 = \left\langle \left(\frac{1}{4\pi} \int_{4\pi} d\Omega \, \widetilde{\Theta}(\hat{\boldsymbol{n}}) \right)^2 \right\rangle \quad . \tag{23}$$

After some manipulation, we can derive that

$$\sigma_{\oplus}^2 = \sigma^2 \rho_0 \quad , \tag{24}$$

which relates the point variance to the variance of global averages. It is convenient to express our result in dimensionless form that allows us to compare the sampling error variance due to the sparseness of the gauges to the variance of the 'natural variability' of global averages. As a kind of 'figure of merit' convenient to the climatologist, we define

$$\Lambda_{\{N\}} \equiv \frac{\sigma_{\oplus}^2}{\epsilon^2} \tag{25}$$

where the subscript $\{N\}$ refers to a specific configuration of N gauges. This measure is a kind of 'signal to noise' for the variances. Clearly we want it to be as large as possible. An equivalent measure is the percent of the total measured variance contributed by sampling error variance:

$$V_{\{N\}} = \frac{\epsilon^2}{\epsilon^2 + \sigma_{\oplus}^2} \cdot 100\% \tag{26}$$

$$= \frac{1}{1 + \Lambda_{\{N\}}} \cdot 100\% \quad . \tag{27}$$

Before turning to specific configurations consider the special case of $N = 1$. We obtain

$$\Lambda_{\{1\}} = \frac{\rho_0}{1-\rho_0} \tag{28}$$

$$V_{\{1\}} = (1-\rho_0)\cdot 100\% \ . \tag{29}$$

This tells us that the error is very sensitive to how much of the variance of the field is concentrated in the global mode. A very 'red' spectrum will yield better estimates with only a few gauges. A very red spectrum means that the ρ_n tend to zero quickly as n increases. This in turn means that the autocorrelation length on the sphere is large.

Clearly the limit $N \to \infty$ is also of interest. It seems likely that $\Lambda_{\{\infty\}} \to \infty$ and $V_{\{N\}} \to 0$. The proof (not attempted here) must make use of the partitioning of the point gauges on the sphere and the convergence properties of the spectrum.

4. Randomly Distributed Gauges Over the Surface of the Sphere

In the last section, we presented a general formula to evaluate the sampling error for deterministically distributed point gauges on a spherical surface. In this section, we discuss the case that the point gauges are randomly distributed. Let $\hat{\boldsymbol{n}}_i$ denote the position of the i-th gauge. Since it is assumed that the gauges are randomly distributed, $\hat{\boldsymbol{n}}_i$ is a random variable and has an associated probability distribution function (pdf). We denote this pdf by $\wp(\hat{\boldsymbol{n}}_i)$ and suppose that it can be expanded in spherical harmonics as

$$\wp(\hat{\boldsymbol{n}}_i) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \wp_n^m(i) Y_n^m(\hat{\boldsymbol{n}}) \ . \tag{30}$$

The simplest possible case is that the distribution of the gauges is uniform. Namely,

$$\wp(\hat{\boldsymbol{n}}_i) = \frac{1}{4\pi} \ , \qquad i = 1, 2, \ldots, N \ . \tag{31}$$

This corresponds to the zeroth order spherical harmonic in (30). We further assume that all the random variables $\{\hat{\boldsymbol{n}}_i\}_{i=1}^{N}$ are independent. Hence,

$$\wp(\hat{\boldsymbol{n}}_1, \ldots, \hat{\boldsymbol{n}}_N) = \wp(\hat{\boldsymbol{n}}_1)\wp(\hat{\boldsymbol{n}}_2)\ldots\wp(\hat{\boldsymbol{n}}_N) \ . \tag{32}$$

By equation (22) derived at the end of the last section, the mean squared error in the case of randomly distributed gauges is a function of the random variables $\{\hat{n}_i\}_{i=1}^N$. Next we wish to evaluate the expectation value E^2 of $\epsilon^2(\hat{n}_1, \ldots, \hat{n}_N)$.

$$E^2 = \underbrace{\int d\Omega_1 \cdots \int d\Omega_N}_{N} \, \epsilon^2(\hat{n}_1, \ldots, \hat{n}_N) \wp(\hat{n}_1, \ldots, \hat{n}_N)$$

$$= \sigma^2 \sum_{l=1}^{\infty} (2l+1)\rho_l \cdot \frac{1}{N^2} \sum_{i,j=1}^{N} I_{ij} \tag{33}$$

where I_{ij} is given by the following integral

$$I_{ij} = \underbrace{\int d\Omega_1 \cdots \int d\Omega_N}_{N} \, P_l(\hat{n}_i \cdot \hat{n}_j) \wp(\hat{n}_1) \wp(\hat{n}_2) \cdots \wp(\hat{n}_N) \ . \tag{34}$$

Since $\wp(\hat{n}_i)$ is a pdf,

$$\int_{4\pi} d\Omega_i \, \wp(\hat{n}_i) = 1 \ , \qquad i = 1, 2, \ldots, N \ .$$

Hence,

$$I_{ij} = \int d\Omega_i \int d\Omega_j \, P_l(\hat{n}_i \cdot \hat{n}_j) \wp(\hat{n}_i) \wp(\hat{n}_j) \ . \tag{35}$$

This leads to

$$E^2 = \sigma^2 \sum_{l=1}^{\infty} (2l+1)\rho_l \cdot \frac{1}{N^2} \left(N + \sum_{\substack{i,j=1 \\ i \neq j}} I_{ij} \right) \ . \tag{36}$$

Insertion of the spherical harmonic expansions (12) and (30), results in

$$E^2 = \frac{\sigma^2}{N^2} \sum_{l=1}^{\infty} (2l+1)\rho_l \left(N + \frac{4\pi}{2l+1} \sum_{m=-l}^{l} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \wp_l^m(i) \wp_l^{m*}(j) \right) \ . \tag{37}$$

The variance 'signal to noise' is then

$$\Lambda_{\{N\}}^{rand} = \frac{\sigma_\oplus^2}{E^2} \ . \tag{38}$$

In the specific case of uniform distribution of the gauges on the sphere,

$$\wp_l^m(i) = \frac{1}{\sqrt{4\pi}} \delta_{m0} \delta_{l0} \ , \qquad i = 1, 2, \ldots, N \ , \tag{39}$$

equation (37) implies

$$E^2 = \sigma^2 \sum_{l=1}^{\infty} \frac{2l+1}{N} \rho_l \tag{40}$$

and

$$\Lambda_{\{N\}}^{rand} = N \Lambda_{\{1\}} \quad . \tag{41}$$

This last is an interesting formula. It states that the sampling error variance incurred on the average for N randomly located gauges is the same as for N independent measurements with a single gauge. Clearly N gauges on the sphere would lead to a larger error because of correlations (the effective number of gauges would be less than N). The process of averaging over all configurations with N gauges appears to reduce the error to that of N independent measurements. This is counterintuitive as one might suspect that as N increases, the sampling error would decrease. However, considering an ensemble average of all possible random configurations, we find that is not the case. Numerical evidence is provided in Hardin, North, and Shen (1992).

5. A MODEL SPECTRUM

In order to proceed to numerical results useful to climatologists, we need a spectrum for the temperature field. Fortunately, a simple model exists that appears to capture the main features of the space-time correlations of the temperature fluctuations. The model is a so-called noise forced energy balance climate model where the forcing function is characterized by broad band noise in both space and time. For an introduction and thorough discussion of the model the reader is referred to North *et al.* (1983), North and Cahalan (1982), Leung and North (1990, 1991) and Kim and North (1991). The general time dependent stochastic model is given by

$$\tau_0 \frac{\partial}{\partial t} \Theta(\hat{\boldsymbol{n}}, t) - \lambda_0^2 \nabla^2 \Theta(\hat{\boldsymbol{n}}, t) + \Theta(\hat{\boldsymbol{n}}, t) = F(\hat{\boldsymbol{n}}, t) \quad , \tag{42}$$

where $\Theta(\hat{\boldsymbol{n}}, t)$ is the local departure of the temperature from its steady state value (that when the noise forcing is switched off); λ_0 is an inherent length scale in the model and τ_0 is an inherent time scale. We are only interested here in the low frequency limit of the process (the more general case will be treated elsewhere; *cf.* Kim and North 1991). We can find the low frequency limiting system by setting $\partial/\partial t \to 0$ in the last equation. This removes τ_0 and leaves only a dependence on the length scale λ_0 (for low frequencies or long time averages this length scale is about $^{15}/_{60}$ times the radius of the earth); $F(\hat{\boldsymbol{n}})$ is a noise forcing function in space — its spatial spectral density is white. Then

$$-\lambda_0^2 \nabla^2 \widetilde{\Theta}(\hat{\boldsymbol{n}}) + \widetilde{\Theta}(\hat{\boldsymbol{n}}) = F(\hat{\boldsymbol{n}}) \tag{43}$$

and

$$\langle F(\hat{\boldsymbol{n}}) F(\hat{\boldsymbol{n}}') \rangle = \sigma_F^2 \delta(\hat{\boldsymbol{n}} - \hat{\boldsymbol{n}}') \ . \tag{44}$$

We note that if the statistics of F are rotationally invariant on the sphere, then so will those of $\widetilde{\Theta}(\hat{\boldsymbol{n}})$, since the differential operator ∇^2 in the last equation is rotationally invariant.

The physical interpretation of the equation is as follows. The governing equation represents the heat budget for an infinitesimal area on the sphere. Heat is spread on the sphere by a diffusion mechanism, represented by the ∇^2 term. It is damped by thermal radiation of heat energy to space, the linear term in $\widetilde{\Theta}(\hat{\boldsymbol{n}})$.

Finally, small imbalances in the heat budget are modelled by the random processes forcing the system, F. These forcing anomalies might be due to cloud fluctuations, eddy heat fluxes, etc. That this model captures the main features of fluctuations of the surface temperature field has been demonstrated fairly convincingly (*cf.* Leung and North 1991; North *et al.* 1983; Kim and North 1991). The latter two studies use a different effective heat capacity over land and ocean and are able to reproduce the land-sea distribution of the forced seasonal cycle as well as the geographical distribution of the second moment statistics, we are not including the space dependence due to land and sea in this simple model. In the low frequency limit, which we are considering here, the distinction between land and sea disappears.

Taking the Fourier-Spherical Harmonic transform of the governing equation we obtain

$$\widetilde{\Theta}_l^m = \frac{F_l^m}{1 + \lambda_0^2 \, l(l+1)} \tag{45}$$

which leads to

$$\rho_l = \frac{\rho_0}{[1 + \lambda_0^2 \, l(l+1)]^2} \tag{46}$$

where ρ_0 is a normalization guaranteeing that $\rho(1) = 1$. Some values of ρ_0 for different length scales are given Table 1. The preferred value of λ_0 is $^{15}/_{60}$ based on simulations.

Given the (low frequency) spatial spectrum for the temperature field above, we are in position to evaluate the MSE for a given array of point gauges, $\{\hat{\boldsymbol{n}}_i\}_{i=1}^N$. From this derivation we may substitute

$$\frac{\rho_0}{[1 + \lambda_0^2 \, l(l+1)]^2} \tag{47}$$

λ_0	ρ_0
$^{10}/_{60}$	0.0276
$^{15}/_{60}$	0.0613
$^{20}/_{60}$	0.1071

Table 1. Values of ρ_0 for various length scale parameters, λ_0.

for ρ_l in the expression for the variance yielding

$$\epsilon^2 = \sigma^2 \sum_{l=1}^{\infty} \frac{(2l+1)\rho_0}{[1+\lambda_0^2\, l(l+1)]^2} \cdot \frac{1}{N^2} \sum_{i,j=1}^{N} P_l(\hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_j) \ , \tag{48}$$

and estimate this expression choosing a cutoff degree L assuming that contributions to the overall variance from degrees larger than L are negligible. In the following examples, L is chosen to be 15 and the expression calculated is:

$$\epsilon^2 = \sigma_{\oplus}^2 \sum_{l=1}^{L=15} \frac{(2l+1)}{[1+\lambda_0^2\, l(l+1)]^2} \cdot \frac{1}{N^2} \sum_{i,j=1}^{N} P_l(\hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_j) \ . \tag{49}$$

6. NUMERICAL EXAMPLES

In order to illustrate the above formula and its use in investigating configurations of gauges we examine several examples. We choose 3 different sample sizes: 40, 140, and 614 gauges. With the networks of size 40 and 140 we construct several "regular" grids on the sphere and place gauges at the intersections of the grid lines. In several cases in the regular array, multiple gauges are degenerately stacked at the poles making the actual number of distinct gauges less than the nominal 40 and 140.

First consider choosing several equally spaced (angularly) latitude and longitude lines. It should make a difference if we choose latitudes that include the poles, since the gauges placed in equal longitudinal angles will be degenerate. For 40 gauges consider 5 latitudes and 8 longitudes and for 140 gauges consider 10 latitudes and 14 longitudes. We will choose to start and end at the poles and to start and end at 80° (North and South) for comparative purposes. Note that while we placed 40 and 140 gauges on the sphere, we have only 26 and 114 uniquely positioned gauges respectively if we choose the pole-to-pole approach. This does not occur in the case where the poles are not represented. We can also flip the order in which we assign these gauges so that we choose 8 latitudes and 5 longitudes and 14 latitudes and

10 longitudes for the two network sizes. Again notice that the pole-to-pole approach results in 32 and 122 uniquely positioned gauges respectively.

Finally, consider the case in which 614 gauges are placed on the sphere with 1 gauge at each of the poles and the remaining 612 gauges placed at every corner of a $10^\circ \times 10^\circ$ box. In Table 2 we provide the calculated values for $\Lambda_{\{N\}}$ and $V_{\{N\}}$ for all of the example gauge configurations in addition to some simple designs for small samples.

N	Gauge layout	$\Lambda_{\{N\}}$	$V_{\{N\}}$ (%)
1	Anywhere	0.070	93.0
2	(90N,0) (45N,0)	0.131	88.4
2	(90N,0) (0,0)	0.148	87.1
2	(90N,0) (45S,0)	0.149	87.0
2	(90N,0) (90S,0)	0.150	87.0
4	(90N,0) (30N,90E) (30S,180) (90S,90W)	0.327	75.4
4	(90N,0) (30S,0) (30S,120E) (30S,120W)	0.344	74.4
4	(90N,0) (0,0) (0,180) (90S,0)	0.345	74.3
4	Vertices of inscribed tetrahedron	0.348	74.2
6	(90N,0) (0,0) (0,90E) (0,90W) (0,180) (90S,0)	0.610	62.1
26	poles plus 3 rings of 8 (pole-to-pole)	1.021	49.5
40	5 rings of 8 (80N-80S)	1.757	36.3
32	poles plus 6 rings of 5 (pole-to-pole)	1.611	38.3
40	8 rings of 5 (80N-80S)	2.813	26.2
114	poles plus 8 rings of 14 (pole-to-pole)	2.281	30.4
140	10 rings of 14 (80N-80S)	2.686	27.1
122	poles plus 12 rings of 10 (pole-to-pole)	5.130	16.3
140	14 rings of 10 (80N-80S)	6.380	13.6
614	Grid corners at every 10°	7.523	11.7

Table 2. Values of $\Lambda_{\{N\}}$ (variance signal to noise) and $V_{\{N\}}$ (percent of measured variance accounted for by error variance) for various configurations.

As we suspected, as the sample size increases the error decreases, but an equally important property makes itself apparent. For a given sample size, there may be a significant discrepancy in the size of the error due to the configuration. Notice, for example, how in the configurations for 40 gauges we can decrease the percentage of measured variance due to error variance from 36.3% to 26.2% by rearranging the gauges. This fact is even more obvious for the configurations of only two gauges. The smaller the opening angle between the two gauges, the larger the error. In our example, we placed the two gauges at opposite poles; however, since we are assuming rotational invariance, we are free to place them on the two intersecting points of *any* line that passes through the center of the sphere. Our choice of latitudes and longitudes in this section is purely illustrative.

7. CONCLUSION

In this paper we presented a spectral formalism for assessing the random sampling errors in the estimation of the globally averaged temperature for an idealized planet using point gauges. A number of simplifications were imposed to make the presentation as simple as possible: 1) We took the statistics of the random field to be rotationally invariant. 2) We dropped all time dependence, essentially working in the low frequency limit. We then proceeded to work out the mean squared error. Two measures of the error were presented: the ratio of variance of global average temperature to the error variance, a kind of variance 'signal to noise'; and the percentage of the measured variance due to the 'sampling error.' We were able to find a simple analytical expression for N gauges distributed randomly (uniformly) on the sphere. We presented a stochastic model of the surface temperature field which can be used to provide a simple parametric form of the spectrum that can be used in the calculations. Using the model spectrum we were able to produce numerical results for regular configurations of gauges on the sphere. For the model spectrum tuned to earth at low temporal frequencies, we found that the percentage of measured variance due to sampling error ranged from 93% for one gauge to 13.6% for a specific configuration of 140 gauges. We made no attempt to optimally weight the readings from individual gauges or to optimally locate the gauges in this paper. We intend to pursue these as well as the time dependent problems in future papers.

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