

## ERRATUM

In Huang NE, Wu M-L, Qu W, Long SR, and Shen SSP, Applications of Hilbert–Huang transform to non-stationary financial time series analysis, *Applied Stochastic Models in Business and Industry* 2003; **19**: 245–268, the author Jin E. Zhang should have been included in the list of authors. The author list should read as follows:

### Applications of Hilbert–Huang transform to non-stationary financial time series analysis

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‡Jin E. Zhang is supported by the Research Grants Council of Hong Kong.

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### SUMMARY

A new method, the Hilbert–Huang Transform (HHT), developed initially for natural and engineering sciences has now been applied to financial data. The HHT method is specially developed for analysing non-linear and non-stationary data. The method consists of two parts: (1) the empirical mode decomposition (EMD), and (2) the Hilbert spectral analysis. The key part of the method is the first step, the EMD, with which any complicated data set can be decomposed into a finite and often small number of intrinsic mode functions (IMF). An IMF is defined here as any function having the same number of zero-crossing and extrema, and also having symmetric envelopes defined by the local maxima, and minima respectively. The IMF also thus admits well-behaved Hilbert transforms. This decomposition method is adaptive, and, therefore, highly efficient. Since the decomposition is based on the local characteristic time scale of the data, it is applicable to non-linear and non-stationary processes. With the Hilbert transform, the IMF yield instantaneous frequencies as functions of time that give sharp identifications of imbedded structures. The final presentation of the results is an energy–frequency–time distribution, which we designate as the Hilbert Spectrum. Comparisons with Wavelet and Fourier analyses show the new method offers much better temporal and frequency resolutions. The EMD is also useful as a filter to extract variability of different scales. In the present application, HHT has been used to examine the changeability of the market, as a measure of volatility of the market. Published in 2003 by John Wiley & Sons, Ltd.

KEY WORDS: Hilbert–Huang transform (HHT); empirical mode decomposition (EMD); financial time series; non-linear; non-stationary; data analysis; Hilbert spectral analysis; volatility; stock price analysis

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Contract/grant sponsor: ONR; contract/grant number: N00014-98-F-0412

Contract/grant sponsor: Ocean Process Program

Contract/grant sponsor: NOAA

Contract/grant sponsor: NASA

## 1. INTRODUCTION

Application of rigorous mathematical statistical methods to financial data can be traced to Bachelier [1], who proposed a Brownian motion model for the Paris stock price. From that time on, there have been many new developments: good summaries of the subsequent progress in various aspects can be found in [2–5], as examples. Many of the new statistical analysis and modelling efforts are built on the variable,  $h_n$ , which is the logarithm of the ratio of consecutive financial variable values defined as

$$h_n = \log \frac{S_n}{S_{n-1}} \quad (1)$$

in which  $S_n$  is the value of the financial variable at the  $n$ th time step. For lack of a proper name, we will here designate it as LRCV, which stands for logarithmic ratio for consecutive values. The advantage of this new variable is obvious: even for a highly non-stationary time history of  $S$ , the LRCV are *seemingly* stationary. This should not be a surprise, for it actually is a differentiation operation of the logarithm value of the variable. Indeed,

$$h_n = \log \frac{S_n}{S_{n-1}} = \log S_n - \log S_{n-1} \quad (2)$$

This operation allows LRCV to satisfy the properties of the classical Wiener processes approximately, but *only* approximately. LRCV is so important that many critical parameters of the financial market are defined in terms of it. For example, the most popular form of volatility, the measure of the variability of a market, is defined as

$$\sigma_n = \left\langle \frac{1}{n-1} \sum_{i=1}^n (h_i - \mu)^2 \right\rangle^{1/2} \quad (3)$$

where  $\mu$  is the mean value of  $h_n$ . As straightforward as it looks, there are many difficulties in this approach: To begin with, LRCV is no longer the price but the logarithm of the price, as observed by Samuelson [6]. More seriously, Shiryayev [4] also pointed out correctly that the volatility is itself volatile; it should be a function of time, and even a random variable. But if the volatility is defined as in Equation (3), it would be a constant. Then we might well ask, how representative is it for the volatility measure?

The main difficulty for the existing statistical financial analysis methods as we see it arises from the basic assumption that, even if the market processes are not stationary, LRCV is stationary. This is a consequence of the fact that most of our statistical tools were developed for stationary processes. So, we have to force the data into the stationary mode, otherwise, we will have no tools to process them. For example, if we give up the stationary assumption, we cannot even define the mean, for mean value is only meaningful if the process is stationary within the interval where the averaging operation is carried out. Without the operation of mean, we would not have standard deviation or many of the statistical measures of a random variable. Then, the statistical landscape would be totally different. But with the stationary assumption, we are neglecting the real changes of the market, and are forced to look at the properties of the data globally in an artificial way. Unfortunately, for most of the financial applications, the local properties are more pertinent, for a financial market is inherently non-stationary. Therefore, to analyse financial data, we cannot just assume the data to be stationary, or carry out simple

pre-whitening differencing operations in one form or the other just to make the data seemingly stationary. We should employ a method designed genuinely for non-stationary processes.

In this paper, we present such a method, the Hilbert–Huang transform (HHT) method, which consists of the two steps: (1) the empirical mode decomposition (EMD), and (2) the Hilbert Spectral Analysis, designed specifically for analysing non-stationary and non-linear time series. We will first give a summary of the method, and then apply it to time-frequency analysis of a financial record from the mortgage market. Then, we will also introduce a new way to measure the variability of the market by using the EMD approach as a filtering technique. It is hoped that this new method, which has found many applications in engineering and science problems, might also find some other unique and useful application in the financial arena. Before getting into the specifics of these new applications, we will first present a brief summary of the new method.

## 2. THE HILBERT–HUANG TRANSFORM

To accommodate the inherent non-linearity and non-stationarity of many data types, we have to utilize the new methods that are designed to accommodate such processes. There are many methods for analysing data from linear non-stationary processes. For example, the spectrogram or the fixed window Fourier spectral analysis for musical and speech signals (see, for example, Oppenheim and Shafer [7]); the Wavelet analysis for image representation and compression (see, for example, Daubechies [8]); the Wagner–Ville distribution for electrical engineering and communication problems (see, for example, Cohen [9] and Flandrin [10]); the Evolutionary spectral analysis (see, for example, Lin and Cai [11]); the Empirical Orthogonal Function expansion for meteorological and oceanographic data explorations (see, for example, Simpson [12]); and other miscellaneous methods such as the least squared estimation of the trend, smoothing by moving mean, and differencing to pre-whiten the data (see, for example, Brockwell and Davis [13]); each has its special merits, but all suffer one flaw or another due to the non-linearity and non-stationarity in the data generating processes as discussed by Huang *et al.* [14]. In this paper, we will only compare the HHT with the straightforward Fourier spectral analysis and the continuous Wavelet analysis to demonstrate ability of the new method.

The HHT was proposed by Huang *et al.* [14, 15]. HHT consists of two parts: (1) The Empirical Mode Decomposition, and (2) the Hilbert Spectral Analysis. The key part of the method is the EMD technique with which any complicated data set can be decomposed into a finite and often small number of intrinsic mode functions (IMF). An IMF is defined as any function having the same number of zero-crossings and extrema, and also having symmetric envelopes defined by the local maxima and minima, respectively. The IMF admits well-behaved Hilbert transforms. This decomposition method is adaptive, and therefore, highly efficient. Since the decomposition is based on the local characteristic time scale of the data, it is applicable to non-linear and non-stationary processes. With the Hilbert transform, the IMF yield instantaneous frequencies as functions of time that give sharp identifications of imbedded structures. The final presentation of the results is an energy–frequency–time distribution, designated as the Hilbert Spectrum. We will give a brief description here to make this paper somewhat self-contained. Interested readers should read the original papers by Huang *et al.* [14, 15], where the method was described in great detail. The following is a brief summary from Huang *et al.* [14]:

For an arbitrary time series,  $X(t)$ , we can always compute its Hilbert transform,  $Y(t)$ , as

$$Y(t) = \frac{1}{\pi} \int_{-P}^P \frac{X(t')}{t-t'} dt' \quad (4)$$

where  $P$  indicates the Cauchy principal value. This transform exists for all functions of  $L^p$ -class (see, for example, Titchmarsh [16]). With this definition,  $X(t)$  and  $Y(t)$  form a complex conjugate pair, so that we can have an analytic signal,  $Z(t)$ , as

$$Z(t) = X(t) + iY(t) = a(t)e^{i\theta(t)} \quad (5)$$

in which

$$\begin{aligned} a(t) &= [X^2(t) + Y^2(t)]^{1/2} \\ \theta(t) &= \arctan \frac{Y(t)}{X(t)} \end{aligned} \quad (6)$$

A description on the Hilbert transform with the emphasis on its mathematical formality can be found in Bendat and Piersol [17, 18], for example. Essentially, Equation (4) defines the Hilbert transform as the convolution of  $X(t)$  with  $1/t$ ; and it therefore emphasizes the local properties of  $X(t)$ , even though the transform is global. In Equation (5), the polar co-ordinate expression further clarifies the local nature of this representation. It is the local fit of an amplitude and phase varying trigonometric function to  $X(t)$ . Even with the Hilbert transform, there is still considerable controversy in defining the instantaneous frequency as

$$\omega(t) = \frac{d\theta(t)}{dt} \quad (7)$$

A detailed discussion and justification are given by Huang *et al.* [14].

Independently, the Hilbert transform has also been applied to study vibration problems and to identify some of the non-linear characteristics through the frequency modulation in a non-linear structure by Worden and Tomlinson [19]. Contrary to the suggestion given by Hahn [18], one should not just take any data, perform a Hilbert transform, find the phase function, and define the instantaneous frequency as the derivative of this phase function. If one follows this path, one would obtain negative frequency, and get frequency values that bear no relationship to the real oscillation of the data. This limitation of the data for the straightforward application of Hilbert transform has rendered the method to be of little practical value. The real advantage of the Hilbert transform only became obvious after Huang *et al.* [14, 15] introduced the EMD method.

The EMD method is a necessary pre-processing of the data before the Hilbert transform can be applied. The EMD will reduce the data into a collection of IMF defined as any function satisfying the following conditions:

- (a) in the whole data set, the number of extrema and the number of zero-crossings must either equal or differ at most by one, and
- (b) at any point, the mean value of the envelope defined by the local maxima and the envelope defined by the local minima is zero.

An IMF, representing a simple oscillatory mode, is a counterpart to the simple harmonic function, but it is much more general. With this definition, one can decompose any function as follows: Identify all the local extrema, and then connect all the local maxima by a cubic spline line as the upper envelope. Repeat the procedure for the local minima to produce the lower

envelope. The upper and lower envelopes should cover all the data between them. The mean of the upper and lower envelopes is then designated as  $m_1$ , and the difference between the data and  $m_1$  is the first ‘Proto-Intrinsic Mode Function’,  $h_1$ , i. e.,

$$X(t) - m_1 = h_1 \quad (8)$$

The procedure is illustrated in Huang *et al.* [14].

Ideally,  $h_1$  should be an IMF, for the construction of  $h_1$  described above should have made it satisfy all the requirements of an IMF. Yet, even if the fitting is perfect, a gentle hump on a slope can be amplified to become a local extremum by changing the local zero from a rectangular to a curvilinear co-ordinate system. After the first round of sifting, the hump may become a local maximum. Therefore, the sifting process should be applied repeatedly.

This sifting process serves two purposes: to eliminate riding waves and to make the wave profiles symmetric. While the first condition is absolutely necessary for separating the intrinsic modes and for defining a meaningful instantaneous frequency, the second condition is also necessary in case the neighbouring wave amplitudes have too large a disparity. Toward these ends, the sifting process has to be repeated as many times as is required to reduce the extracted signal from a ‘Proto-IMF’ to an IMF. In the subsequent sifting processes,  $h_1$  is treated as the data, and then

$$h_1 - m_{11} = h_{11} \quad (9)$$

Repeated siftings, up to  $k$  times, yield

$$h_{1(k-1)} - m_{1k} = h_{1k} \quad (10)$$

and  $h_{1k}$  becomes an IMF. It is designated as

$$c_1 = h_{1k} \quad (11)$$

the first IMF component from the data.

Overall,  $c_1$  should contain the finest scale or the shortest period component of the signal. We can separate  $c_1$  from the rest of the data by

$$X(t) - c_1 = r_1 \quad (12)$$

Since the residue,  $r_1$ , still contains longer period components, it is treated as the new data and subjected to the same sifting process as described above. This procedure can be repeated to obtain all the subsequent  $r_j$ 's, and the result is

$$\begin{aligned} r_1 - c_2 &= r_2 \\ &\dots \\ r_{n-1} - c_n &= r_n \end{aligned} \quad (13)$$

The sifting process will end finally when the residue,  $r_n$ , becomes a constant, a monotonic function, or a function with only one maximum and one minimum from which no more IMF can be extracted. Even for data with zero mean, the final residue still can be different from zero. If the data have a trend, the final residue should be that trend. By summing up Equations (12) and (13), we finally obtain

$$X(t) = \sum_{j=1}^n c_j + r_n \quad (14)$$

Thus, we achieve a decomposition of the data into  $n$ -empirical modes, and a residue,  $r_n$ , which can be either the mean trend or a constant. The components of the EMD are usually physically meaningful, for the characteristic scales are defined by the physical data. Additionally, we can also identify a new usage of the IMF components: filtering. Traditionally, filtering is carried out in frequency space only. But there is a great difficulty in applying the established frequency filtering techniques when the data is either non-linear or non-stationary, or both, for both non-linear and non-stationary data generate harmonics of all ranges with these earlier methods. Therefore, any filtering with these original methods will eliminate some of the harmonics, which will cause deformation of the wave forms of the fundamental modes even if they are outside of the filtering range. Using IMF, however, we can devise a time-space filtering approach. For example, a low pass filtered result of a signal having  $n$ -IMF components can be simply expressed as

$$X_{lk}(t) = \sum_k^n c_j + r_n \quad (15)$$

and high pass results can be expressed as

$$X_{hk}(t) = \sum_1^k c_j \quad (16)$$

further, a band pass result can be expressed as

$$X_{bk}(t) = \sum_b^k c_j \quad (17)$$

The advantage of this time-space filtering is that the results preserve the full non-linearity and non-stationarity in physical space.

Contrary to almost all other earlier methods, this new method is intuitive and direct, its basis is *a posteriori* and also adaptive, which mean it is based on and derived from the data. The decomposition steps are based on the simple assumption that any data consists of different simple intrinsic modes of oscillations. Each mode may or may not be linear, and will have the same number of extrema and zero-crossings. Furthermore, the oscillation will also be symmetric with respect to the 'local mean'. At any given time, the data may have many different coexisting modes of oscillation, one superimposing on the others. The result is the final complicated data. Each of these oscillatory modes is represented by an IMF.

Having obtained the IMF components, one will have no difficulty in applying the Hilbert transform to each IMF component, and computing the instantaneous frequency according to Equation (7). After performing the Hilbert transform on each IMF component, the original data can be expressed as the real part, RP, in the following form:

$$X(t) = RP \sum_{j=1}^n a_j(t) e^{i \int \omega_j(t) dt} \quad (18)$$

Here we have left out the residue,  $r_n$ , on purpose, for it is either a monotonic function, or a constant. Although the Hilbert transform can treat the monotonic trend as part of a longer oscillation, the energy involved in the residual trend representing a mean offset could be overpowering. In consideration of the uncertainty of the longer trend, and in the interest of the information contained in the other low energy but clearly oscillatory components, the final

non-IMF component should be left out. It could, however, be included, but only if physical considerations justify its inclusion. The energy of the signal given in Equation (18) presented in time–energy–frequency space as  $H(\omega, t)$  is designed as the Hilbert spectrum.

Equation (18) gives both amplitude and frequency of each component as functions of time. The same data if expanded in a Fourier representation would be

$$X(t) = RP \sum_{j=1}^{\infty} a_j e^{i\omega_j t} \quad (19)$$

with both  $a_j$  and  $\omega_j$  as constants. The contrast between Equations (18) and (19) is clear: The IMF represents a generalized Fourier expansion. The variable amplitude and the instantaneous frequency have not only greatly improved the efficiency of the expansion, but also enabled the expansion to accommodate non-linear and non-stationary data. With the IMF expansion, the amplitude and frequency modulations are also clearly separated. Thus, we have broken through the previous restriction of the constant amplitude and fixed frequency Fourier expansion, and arrived at a variable amplitude and frequency representation. This frequency–time distribution of the amplitude is designated as the Hilbert amplitude spectrum,  $H(\omega, t)$ , or simply the Hilbert spectrum. If amplitude squared is more preferred (to represent energy density), then the squared values of amplitude can be substituted to produce the Hilbert Energy Spectrum just as well.

The skeleton Hilbert spectrum presentation is more desirable, for it gives more quantitative results. Actually, Bacry *et al.* [20] and Carmona *et al.* [21] have tried to extract the Wavelet skeleton as the local maximum of the wavelet coefficient. Even that approach is still encumbered by the harmonics. If more qualitative results are desired, a ‘fuzzy’ or ‘smeared’ view can also be derived from the skeleton presentation by using two-dimensional smoothing.

With the Hilbert spectrum defined, we can also define the marginal spectrum,  $h(\omega)$ , as

$$h(\omega) = \int_0^t H(\omega, t) dt \quad (20)$$

The marginal spectrum offers a measure of total amplitude (or energy) contribution from each frequency value. It represents the cumulated amplitude over the entire data span in a probabilistic sense.

The combination of the EMD and the Hilbert Spectral Analysis is designated as the Hilbert–Huang Transform (HHT).

### 3. TIME–FREQUENCY ANALYSIS OF THE WEEKLY MORTGAGE RATE DATA

Having described the method, we will present an example to illustrate the differences among the Fourier, Wavelet and Hilbert spectral analyses. The data used here is the weekly mean of the thirty year mortgage rate covering the period from January 1972 to December 2000. The raw data are shown in Figure 1. This data set, after subjected to the EMD, yields eight IMF components shown in Figure 2. Here we can immediately see many interesting features of the data from just the IMF components. To begin with, there is an obvious change in the data quality starting around 1980, when the amplitudes of the short period IMF components (i.e. c1 and c2) suddenly increase. Can this change in short period components be interpreted as volatility? We think so, and we will return to this point later. Second, there is a large amplitude, long period IMF component seen in c6 with a period of approximately 8 years. More details of



Figure 1. The raw data of weekly averaged quotations for the 30-year mortgage rate covering the period January 1972 to December 2000.

the properties from the IMF will have to wait until we construct the Hilbert Spectrum. Now let us examine the meaning of and utility for the IMF from a different point of view: the reconstruction of the data.

To demonstrate the intrinsic meaning of these IMF components, we will reconstitute the data from the components. The sequence of steps is shown as follows in Figure 3: In each of the sub-panels, we plot the data as a dotted line and the partial sum of the IMFs as a solid line. In Figure 3(a), we plot the data and component  $c_8$ , the residue of the sifting. As we can see from Figure 2, the slope of the residue term is very small, about 0.05% over the total period. This certainly is not a significant trend, but it establishes global level of the data. It should be noted that the residue term is not the mean, for it is not derived from averaging processes. Rather, it is the residue after all possible oscillations are removed by the EMD steps. To this residual trend, if we add the longest oscillatory component,  $c_7$ , we have the result in Figure 3(b). This smooth line clearly gives the smoothest trend of the data variation. With step by step adding of the IMF components, we finally arrived at the sum of all the IMF components shown in Figure 3(h). It looks like the original data. In fact, the difference between the total sum of IMFs and the original data is of the order of  $(10^{-5})$

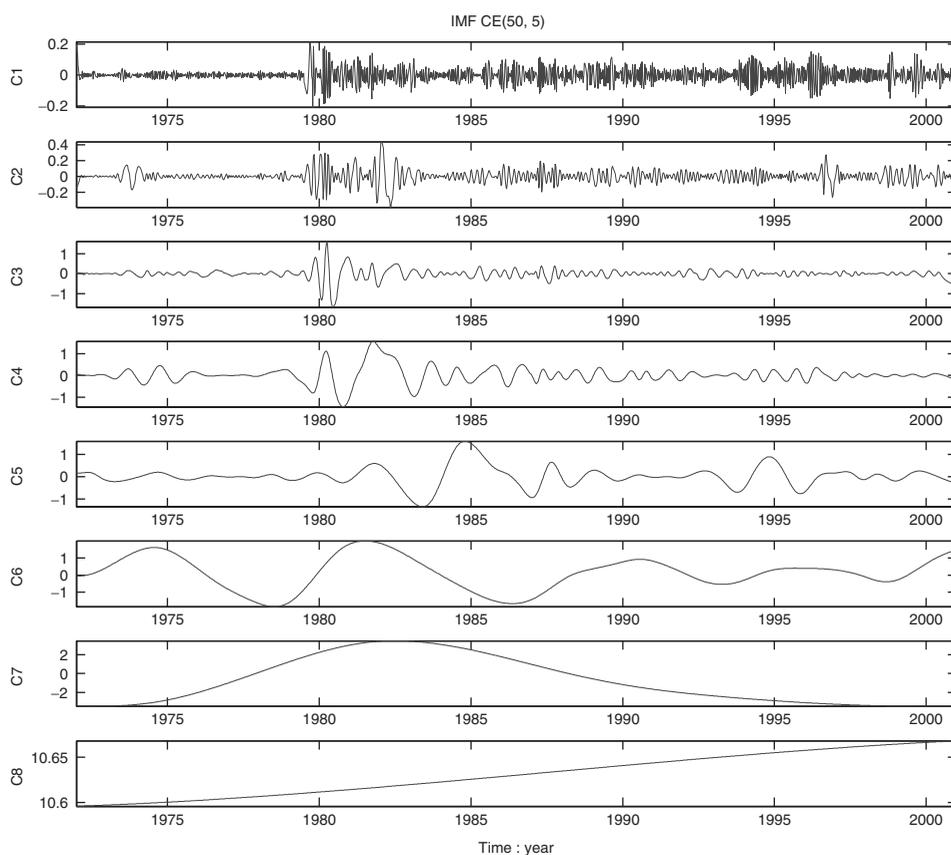


Figure 2. The IMF for the data shown in Figure 1 through the Empirical Mode Decomposition method using extrema-based sifting.

shown in Figure 4. As the data are only kept to the third place decimal, this difference is the round-off error in the computation. The completeness of the decomposition is thus demonstrated.

This reconstruction procedure also illustrates the use of the EMD process as a filter, as shown in Equations (15) to (17). In the present case, it is the low pass filtering. If we stop at any step, we would have the market trend with a time scale longer than the characteristic period of the next IMF component not included. Note that this time scale is defined by the data, rather than by a pre-assigned value, a truly crucial difference between EMD filtering and the Fourier analysis based filtering. Additionally, this filtering is non-linear. Unlike the Fourier operation in frequency space, it would not eliminate any harmonics of the fundamentals required by the non-stationarity and non-linearity of the data. We will return to this point later in the paper when we discuss the high pass application in defining the volatility. In the rest of this section, we will concentrate on the study of the time–frequency characteristics of the data. For this purpose, we need the Hilbert Spectrum.

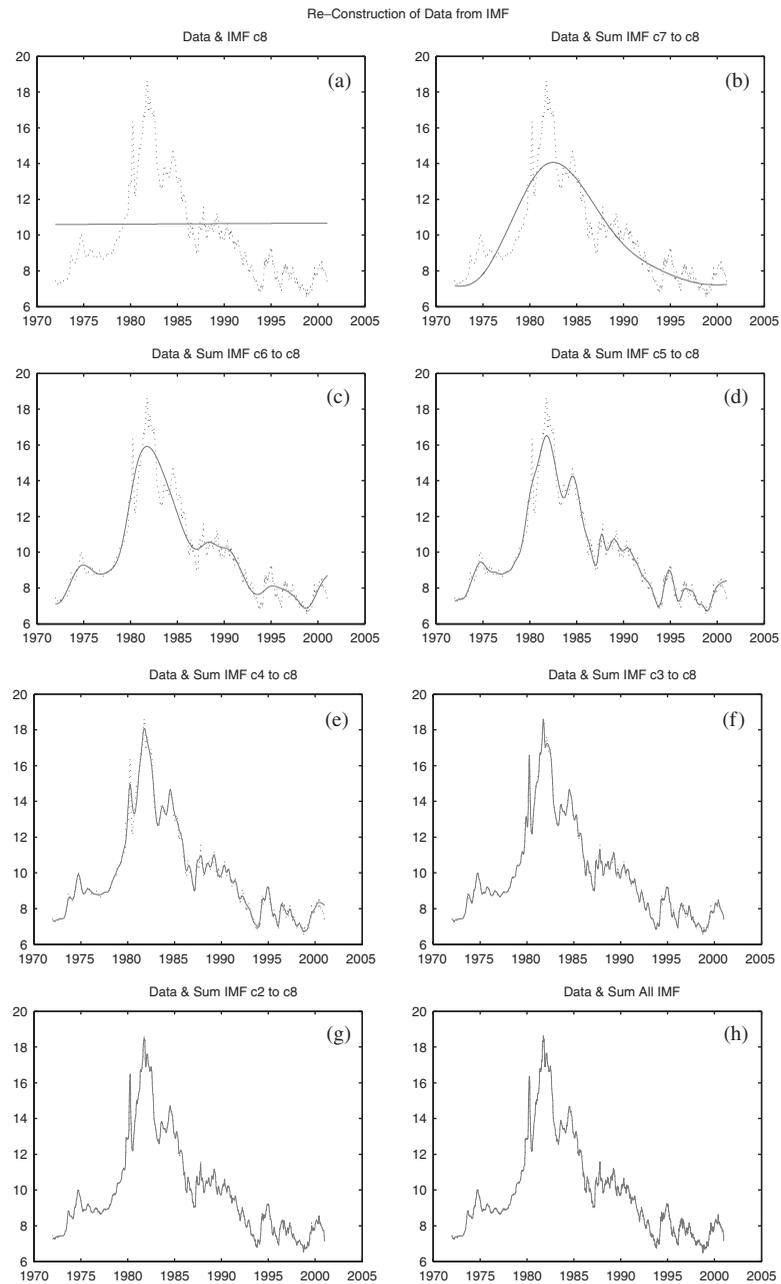


Figure 3. Re-construction of the data from the IMF components. This is also a demonstration for using the EMD technique as a filter.

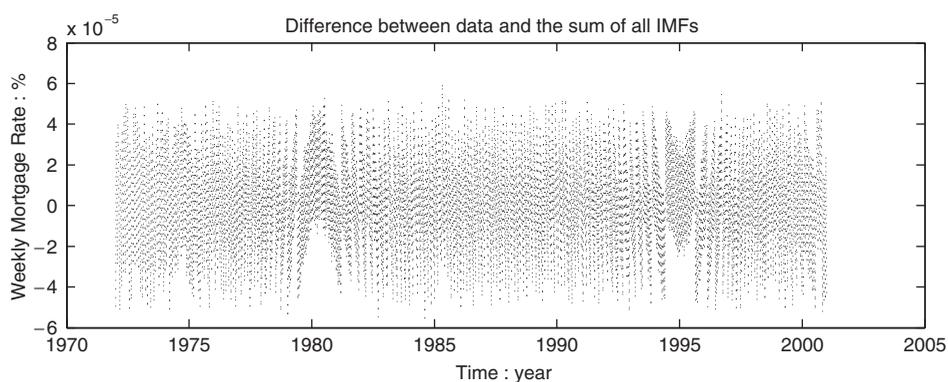


Figure 4. The difference between the data and the reconstructed time series from IMF. The difference is of the order of the round-off error.

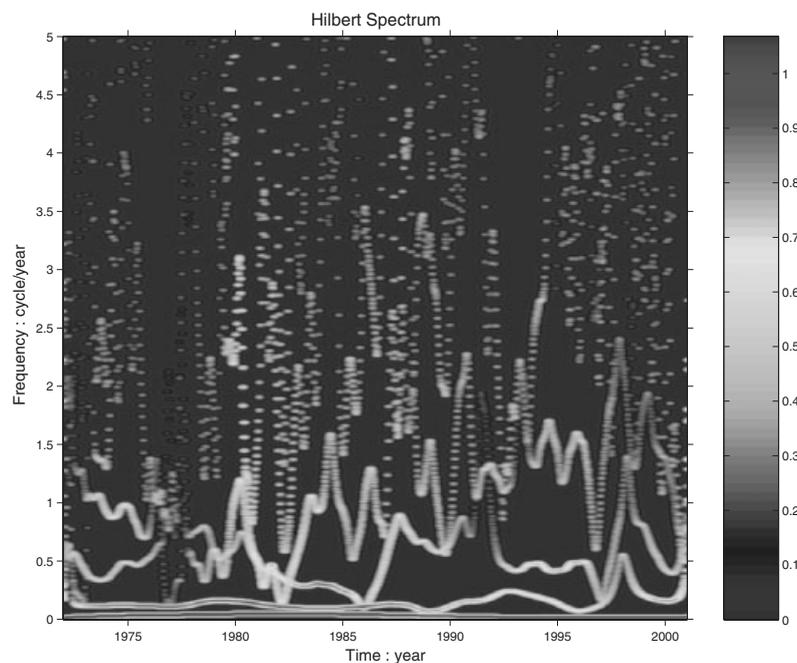


Figure 5. The Hilbert spectrum based on the IMF given in Figure 2.

Following the steps given in Equation (18), we can construct the Hilbert spectrum as shown in Figure 5. From the same data, we can also compute the Wavelet spectrum. In order to extract data variation features, we have used the continuous Morlet Wavelet transform, and the result is given in Figure 6. When these two results are placed side by side, they bear little resemblance to

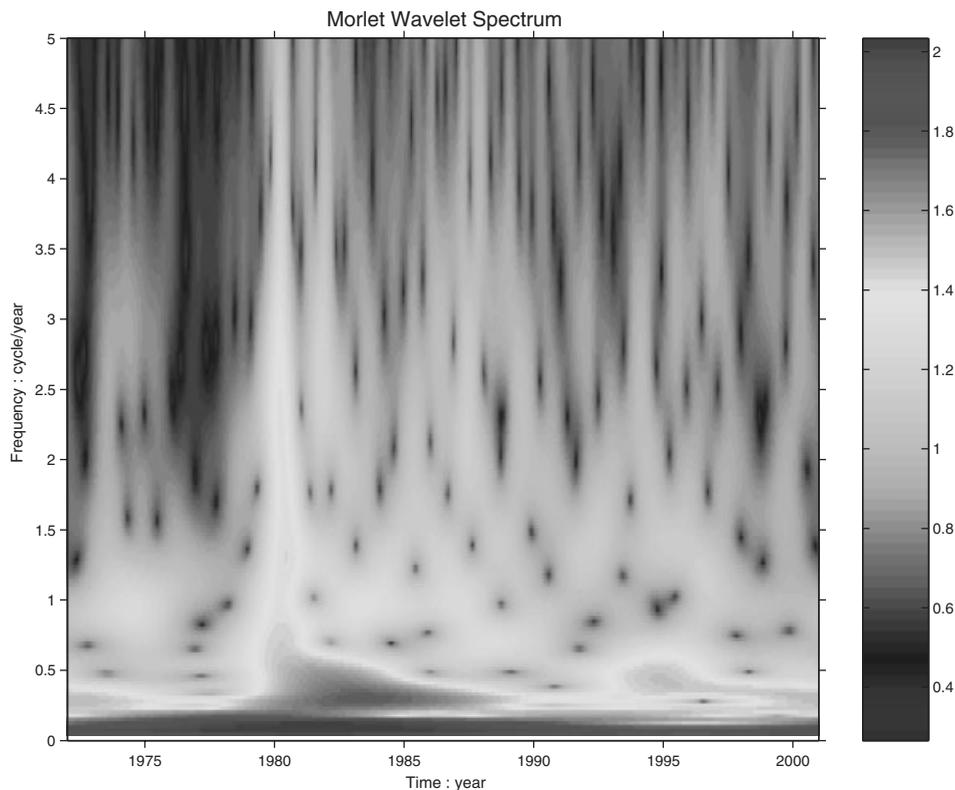


Figure 6. The Morlet Wavelet spectrum based on the data given in Figure 1.

the first impression. Only through careful scrutiny, will one see that they are not inconsistent with each other. In fact, the Hilbert spectrum is almost the skeleton form of the Wavelet spectrum as proposed by Carmona *et al.* [21], but there are crucial differences: As the EMD is a non-linear decomposition, the energy distribution in the Hilbert spectrum contains all the non-linear waveform distortion, such as intra-wave frequency modulations Huang *et al.* [14]. This non-linear representation does not need the harmonics to fit the waveform. Furthermore, the frequency determined by differentiation is precise in both time and frequency values; it breaks through the limitation of the uncertainty principle (see, for example, Cohen [9] and Flandrin [10]) inherited in the Fourier transform pairs, or Fourier type of transform pairs such as the Wavelet transform.

But there are also some similarities between the Wavelet and HHT results, of course. To illustrate some of the similarities, we have smoothed the Hilbert spectrum with an  $11 \times 11$  Laplacian filter applied repeatedly for six times, to really 'smear out' the result. This is given in Figure 7. Now, one can see the general agreement of the energy distributions. To go one step further, we also plotted the energy distribution contours of the smoothed Hilbert spectrum on the Wavelet spectrum in Figure 8. Here, the agreement and disagreement are

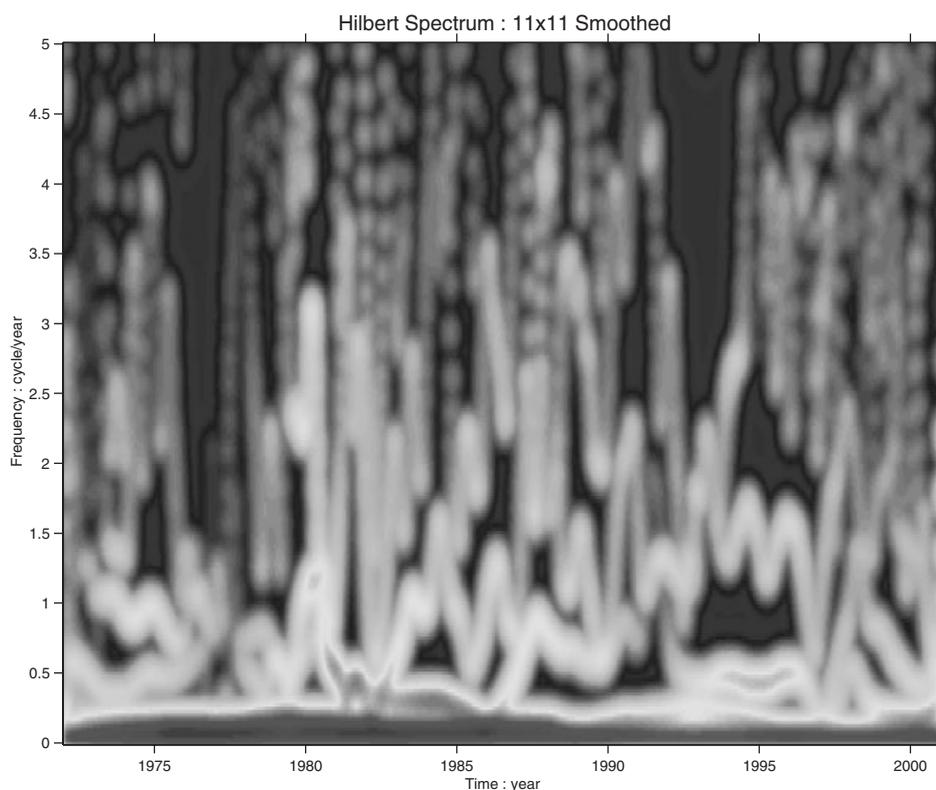


Figure 7. The smoothed Hilbert spectrum using an  $11 \times 11$  Gaussian weighted Laplacian filter repeated six times.

all clear: The gross energy distributions are the same in both, but there are no long streaks of energy concentration covering all frequency ranges at any given time in the Hilbert spectrum as is found in the Wavelet spectrum (the consequence of the harmonics). The results given in the last four figures emphasize the time–frequency–energy resolution power of this different method in the time–frequency space. To further illustrate the frequency resolution power of the Hilbert spectral analysis, we plot the marginal spectra of Hilbert and Wavelet spectra together with the Fourier spectrum in Figure 9, with their magnitude staggered by a decade to show the individual spectral characteristics. The extreme redundancy and the uniform but poor frequency resolution of the Wavelet spectrum are clear now. With 5.5 waves in the basic Wavelet, the Morlet Wavelet gives the poorest overall frequency resolution. Although it does resolve the energy variation in time, the result seems to be only qualitative; therefore, it should not be used as a time–frequency analysis tool, as discussed in Flandrin [10] and Huang *et al.* [22]. The non-uniform frequency resolution of the Fourier spectrum, on the other hand, does a good job in extracting some frequency bands, such as the location of the energy concentration, even though it does not bear any information on the time axis. We believe that we have demonstrated that both the Fourier and Wavelet representations are inferior to the Hilbert result, which gives

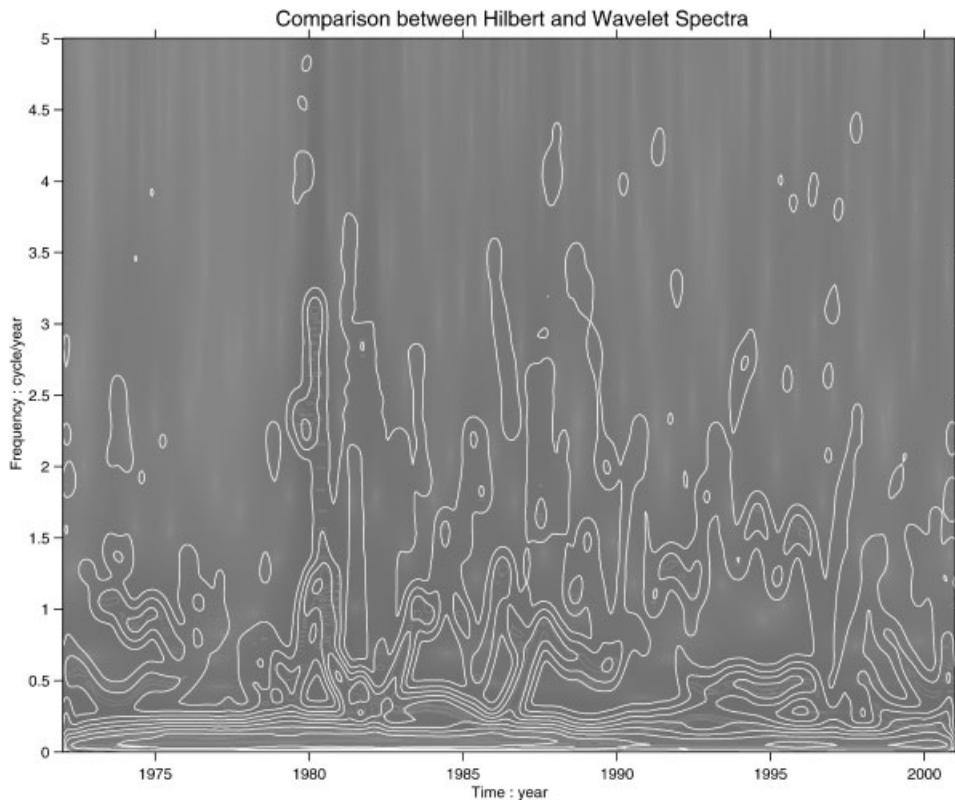


Figure 8. The comparison between the Morlet Wavelet and the Hilbert spectra. The Wavelet spectral values are in colour intensity, and the smoothed Hilbert values are given in contours.

a much more detailed frequency and time resolution. Let us discuss the variation of the data in frequency space in more detail.

Due to the poor frequency resolution of the Wavelet spectrum, it will not be discussed any further. The Fourier spectrum, however, shows some interesting energy concentrations: For example, the peak at around 0.3 cycle/year coincides with the one also in the Hilbert spectrum. It represents a period of slightly more than 3 years. There should be energy at frequency bands lower than this peak, but with the limited length of the data, Fourier cannot resolve any peak beyond this value, except to show a general upward hump ending at 0.034 cycle/year, the theoretical limit of the Fourier analysis with the given data. The Hilbert spectrum, however, gives spectral values all the way down below 0.01 cycle/year. The broad spectrum covering 0.02 to 0.05 cycle/year represents the period of the full data length. Whether one can or cannot treat this as a genuine oscillatory component is debatable. But the data certainly suggests such a period, and the Hilbert spectral analysis correctly identifies it with only one cycle over the entire data span, a feat unmatched by any other known data analysis method in existence. The peak at around 0.12 cycle/year is an

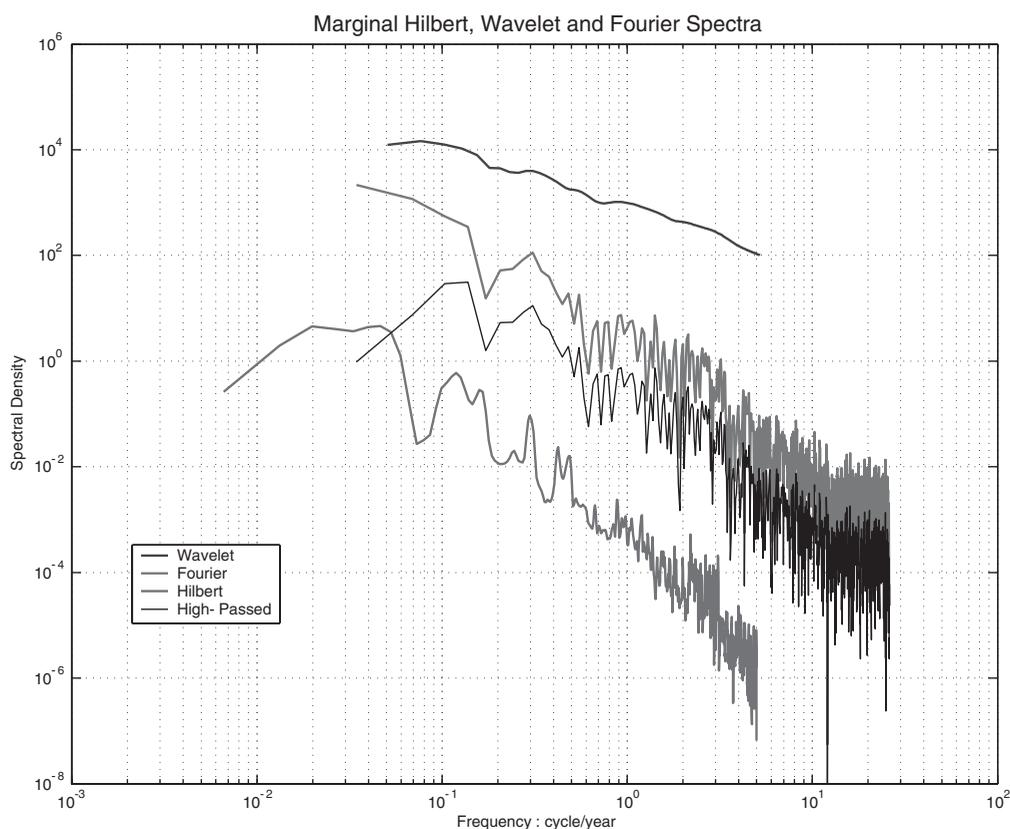


Figure 9. The inter-comparison of the Marginal Hilbert and Wavelet spectra with the Fourier spectrum. Also shown is the Fourier spectrum based on the EMD high-pass filtered data.

8 year period, which might be significant, for it is the usual tenure of the US Presidents. With this as a guide, we can re-examine the peak at more than 3 years in both the Fourier and Hilbert spectra. It is very close to, and therefore could also be, the 4 year general election cycle. Another location of energy concentration is around 2 to 3 cycle/year, but this peak is diffused and with much lower magnitude; therefore, it might not be significant. Now, let us return to the application using the EMD as a filter.

In the last section, we have discussed the use of the EMD technique as a filter. If we apply the high pass filter by summing only the first six IMF components, we would get the result shown in Figure 10. The Fourier spectrum of this data is also given in Figure 9 in the thin black line. Here the 0.12 cycle/year (or the eight year period) peak is clearly shown. In applying the EMD-based filter, we only have to eliminate the component with the longest period, the component covering the whole data span as a single cycle of oscillation. The next component actually represents the 8-year peak. This filtering result cannot be derived with Fourier analysis without a pre-determined low cut-off frequency. So the lower end can only represent the

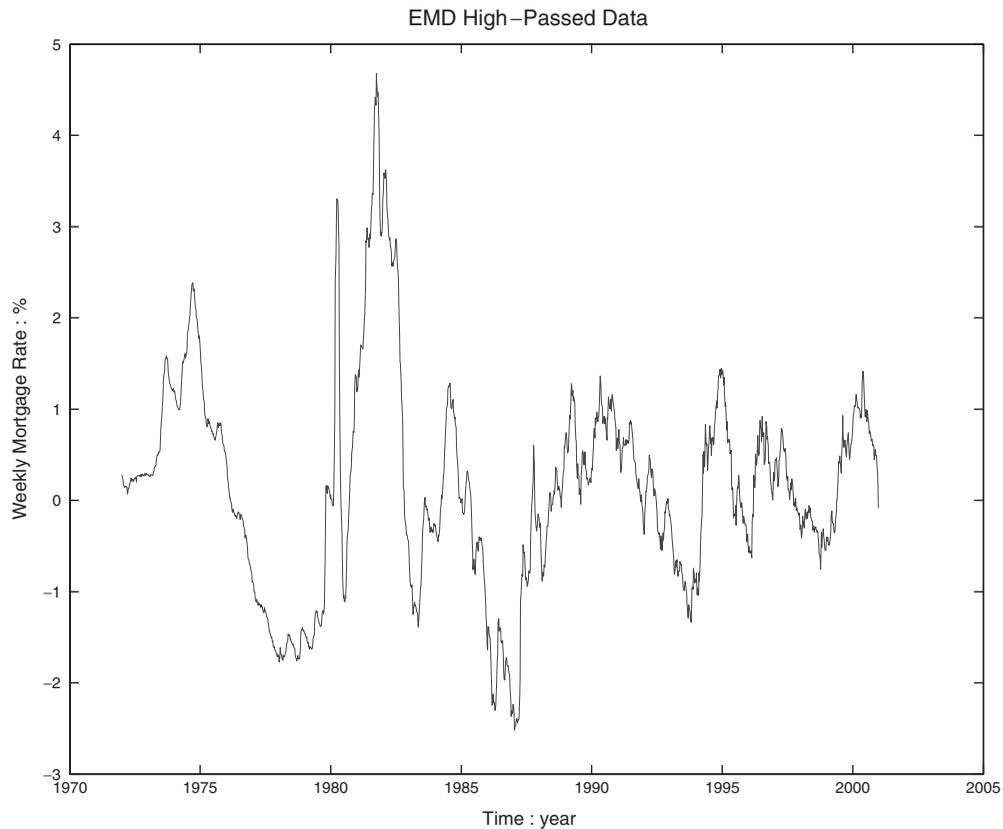


Figure 10. The result of the EMD high-passed filtered data.

lower *a priori* determined cut-off frequency. As a result, the peak in such a Fourier-filtered spectrum, even if it shows up, would hardly be significant, for the selection of the cut-off frequency is judgmental. The peak derived from the EMD-based filter in the figure, however, is meaningful.

#### 4. VOLATILITY

Having examined the data in time–frequency space, we would like to apply the EMD technique in still another application, to address the question of volatility. As stated by Shiryaev [4], the volatility as defined in Equation (3) is arguably the most loosely interpreted financial variable. This is obvious, for in the definition we need the mean and the standard deviation of LRCV. To implement it, we have to assume stationarity. If we compute the LRCV from the data, the result is given in Figure 11. Even casual inspection reveals the non-uniform characteristics of the time series. This change in characteristics is similar to, but not as clear as, what we have shown in Figure 2. Thus we must ask: How can one justify the assignment

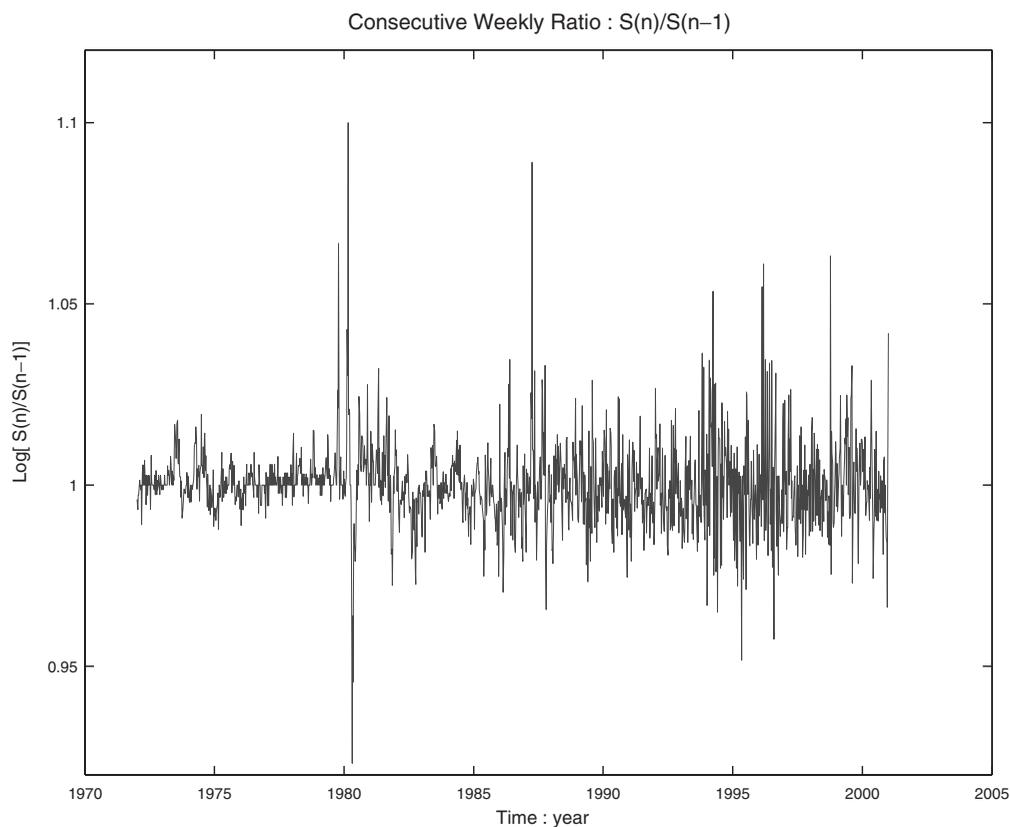


Figure 11. The values of the Logarithm Ratio of Consecutive Value (LRCV), a popular proxy to present the market variability. Notice that the data is not stationary.

of one volatility value to represent the whole time series? If we give up using one value to represent all, how should one divide the time span into sub-periods? How can we be sure that a time series is stationary within each sub-period? All of these uncertainties have contributed to the looseness of the previous definition of volatility. In fact, if volatility should be a function of time as suggested by Shiryaev [4], it should also have a time scale over which it is applicable. With these observations, we decided to introduce a new measure of volatility based on the EMD-produced IMF components. In order not to be confused with the traditional volatility, we will designate this new measure of volatility as ‘variability’. The variability is defined as the ratio of the absolute value of the IMF component(s) to the signal at any time:

$$V(t; T) = \frac{S_h(t)}{S(t)} \quad (21)$$

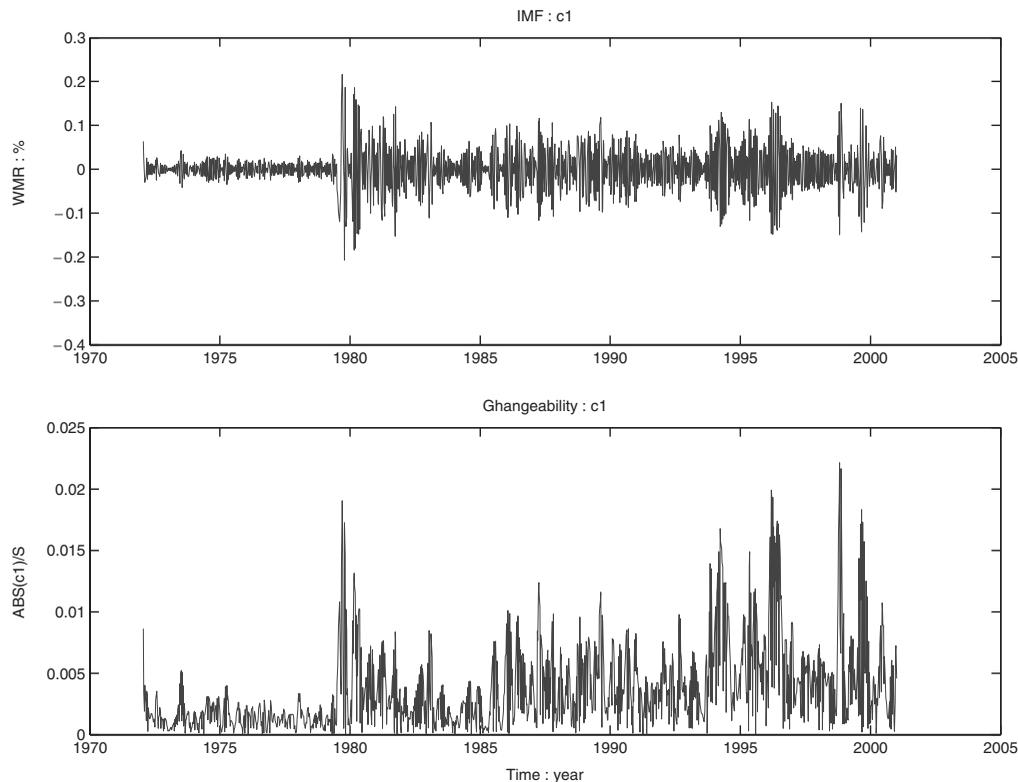


Figure 12. The variability of the weekly mortgage quotations based on the first IMF component having a monthly time scale.

where  $T$  correspond to the period at the Hilbert spectrum peak of the high passed signal up to  $h$ -terms,

$$S_h(t) = \sum_{j=1}^h c_j(t)$$

Therefore, the resulting variability is a function of time as suggested by Shiryayev [4], and additionally,  $h$  shows over what time scale it is applicable. The unit of this variability parameter is the fraction of the market value. It is a simple and direct measure of the market volatility.

Let us take IMF  $c_1$  as an example. The time series of  $c_1$  is shown in the top panel of Figure 12. The change of variability is obvious. This variability measures the change with respect to the local mean, but we achieve it without invoking an averaging process, but instead through the EMD approach. Again without invoking averaging, instead of computing the standard deviation, we simply rectified it by taking the absolute value and normalized

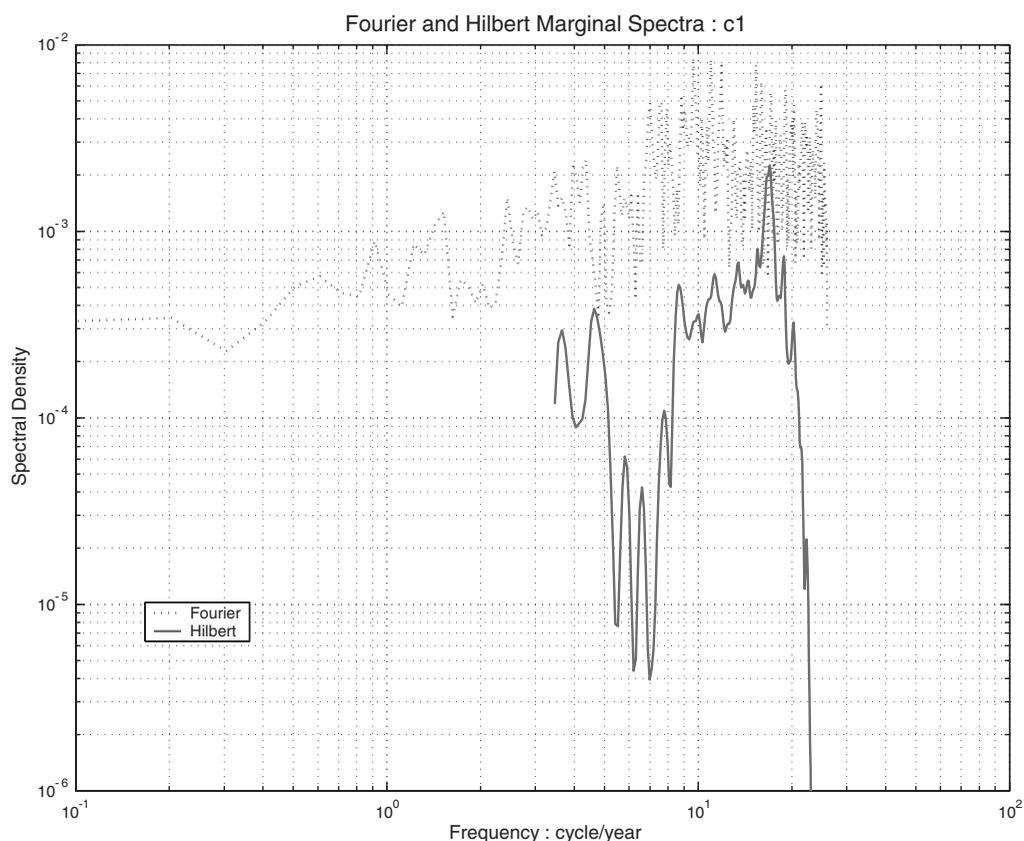


Figure 13. The marginal Hilbert spectrum based on the IMF c1, to determine the time scale for the variability in Figure 12.

it by the instantaneous values of the signal there. This variability is a function of time, with fluctuating values. The amplitude increases drastically after 1985. The overall mean of this variability, though not very meaningful, is about 0.38%; its envelope mean is approximately 0.50%. The mean time scale associated with this variability is defined by the Hilbert spectrum of the IMF c1, as shown in Figure 13. Roughly it is about a monthly period. This again utilizes the filtered results of the EMD. This EMD filter is totally adaptive, and applicable to non-linear and non-stationary processes. Precisely because of the non-linear and non-stationary characteristics of the EMD filter, this kind of filtering is not amenable to a Fourier approach, for the Fourier filter should only be applied to linear and stationary processes. If one uses a Fourier-based monthly frequency band to filter the data, two complications will arise: First, the non-linear harmonic distortion will cause leakage of energy from the low frequency fundamentals to a higher frequency range of the result. Secondly, to fit a non-stationary time series with constant amplitude and frequency, the sinusoidal functions will require a much wider range of frequency. This will cause energy to leak out of the

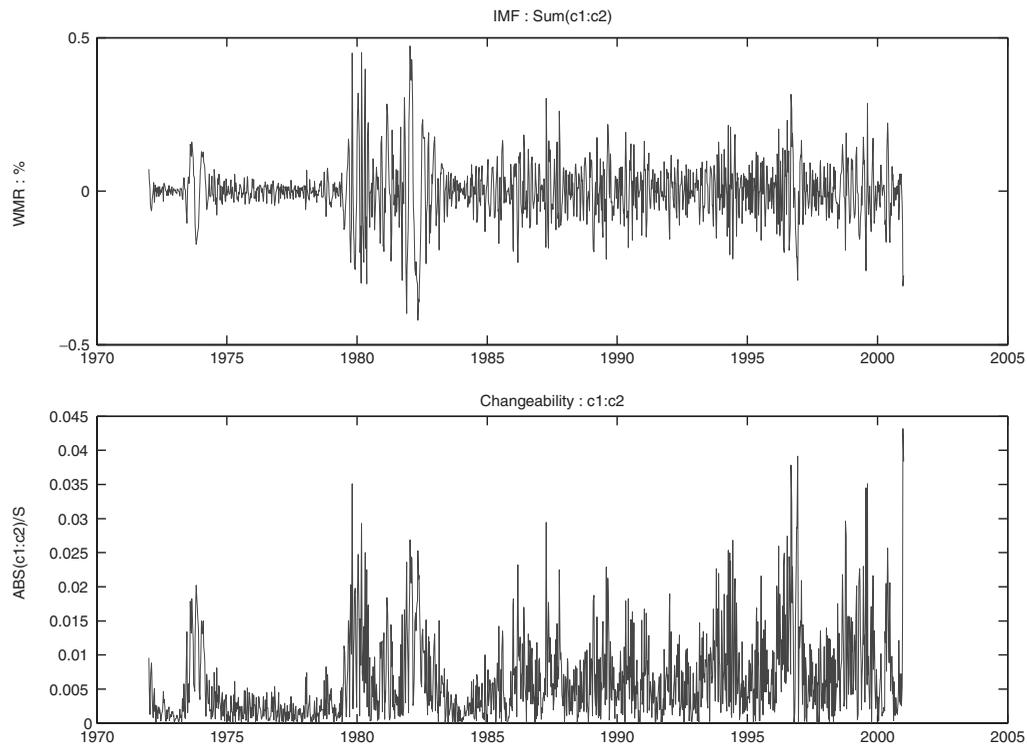


Figure 14. The variability of the weekly mortgage quotations based on the first and second IMF components having a semi-annual time scale.

designated range as shown in Figure 13. The combined effects of the above shortcomings will make the final result unreliable.

If one wants to measure the variability over a longer period, one can sum more than one IMF and get the combined results. Such results are shown in Figures 14 and 15. In the upper panel of Figure 14, we have the data as the sum of IMFs  $c_1$  and  $c_2$ . In the lower panel we have the overall mean variability values increased to around 0.63%; the envelope mean is 0.86%. The combined IMF components give a marginal Hilbert spectrum peaked around 2 cycles/year, or with a half-yearly period. The combination of the first three IMFs gives the data and variability in Figure 16. Now the variability is much higher. The values for overall and envelope means are 1.48 and 2.09%, respectively. Its corresponding period increases to around a year as shown in Figure 17.

Thus, we have defined a variability value referenced to the local mean, as a function of time, and also associated with a frequency range over which it is applicable. This new definition can offer a direct measure of the market value in percentages rather than the logarithm of the market value. It certainly provides another meaningful measure of volatility.

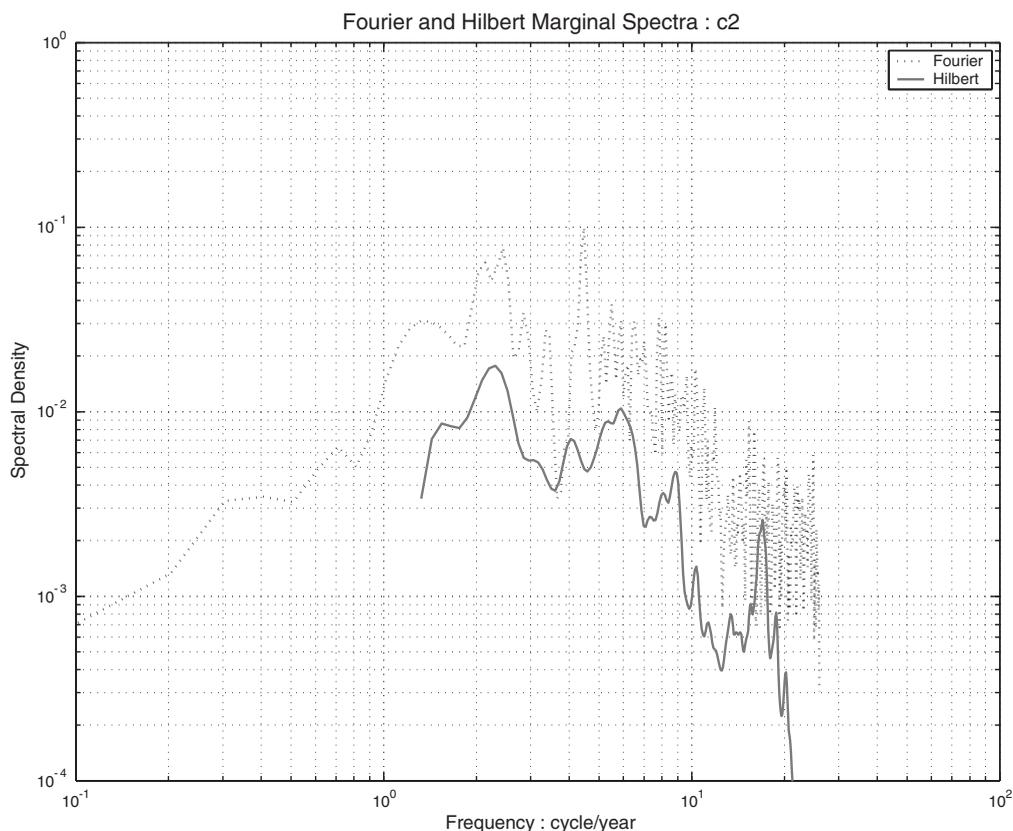


Figure 15. The marginal Hilbert spectrum based on the sum of the IMFs  $c_1$  and  $c_2$  to determine the time scale for the variability in Figure 14.

## 5. DISCUSSIONS

We have introduced a new statistical analysis tool for non-linear and non-stationary data. It gives results not only in time–frequency space, but also provides a detailed time decomposition of the data. There is one further note on the frequency definition: As pointed out by Flandrin [10], when the signal is non-linear, one cannot assume that the Hilbert transform of the signal will have precisely the same phase function as the real part with only a phase shift. Under such a condition, the instantaneous frequency might not be exactly the differentiation of the phase function of the real part. An example of this has been given by Huang *et al.* [14]. Therefore, in application, we have always checked the frequency of the fundamental modes with either Fourier or Wavelet analyses to guarantee the fidelity of the results. A necessary condition is for the data to satisfy the Bedrosian theorem (see, for example, Hahn [18]). It should also be pointed out that the Hilbert transform is not the only way to compute the instantaneous frequency, Potamianos and Maragos [23] have proposed an energy operator to compute the instantaneous frequency directly from the

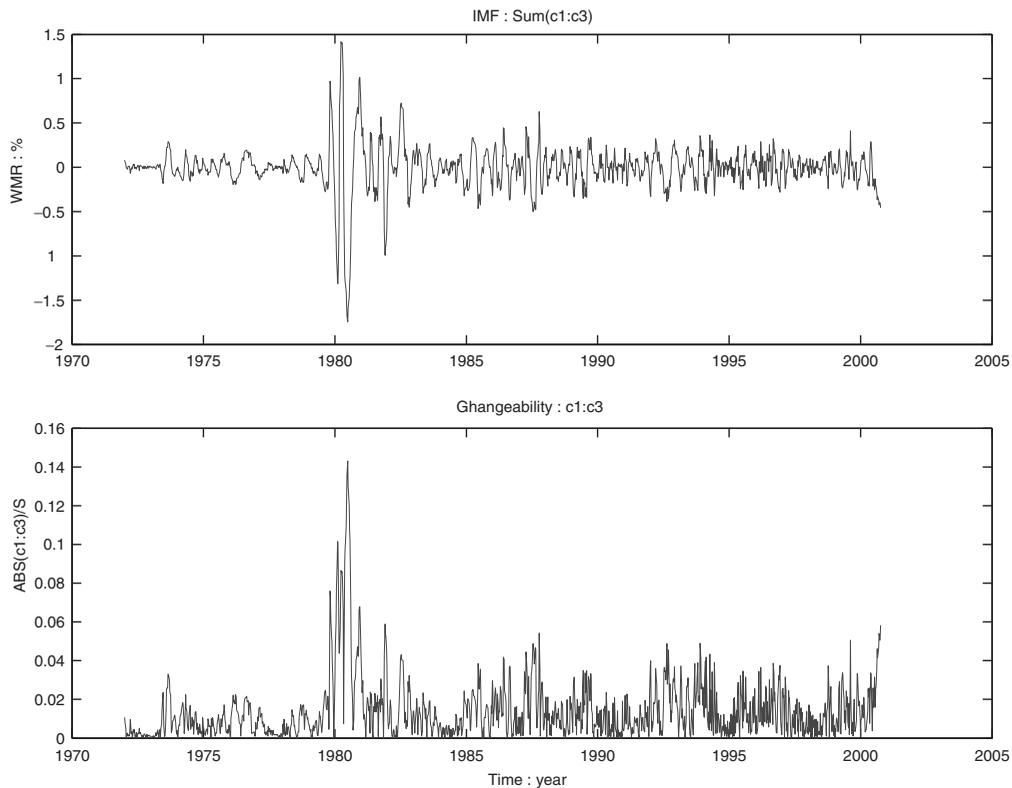


Figure 16. The variability of the weekly mortgage quotations based on the first three IMF components having an annual time scale.

signal independent of the Hilbert transform. In most of the examples studied by them, the results are almost identical to the values obtained through the Hilbert transform. The frequency is certainly an important parameter to be derived from the data, but we have also demonstrated the wide utilities of the IMF in defining the variability, such as the volatility, as a basis for data expansion, and for filtering in temporal space. Therefore, the EMD method by itself is also a useful tool in statistical analysis of non-linear and non-stationary data.

## 6. CONCLUSIONS

We have introduced here a new statistical method for financial data analysis. As most of the financial data are inherently non-stationary and non-linear, it is important that we adopt a method designed for such processes. The insouciant assumptions of stationary and homogeneous steps and many other similar ones need careful scrutiny. We feel that this new method deserves a trial in this new area of financial data analysis.

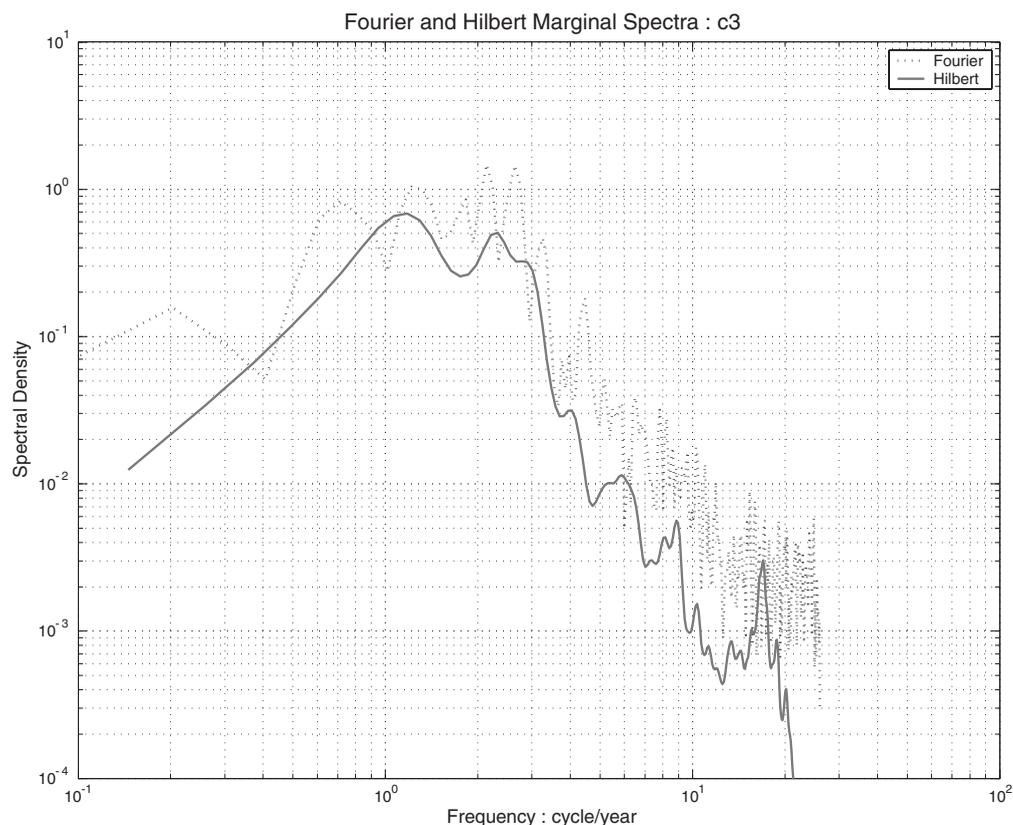


Figure 17. The marginal Hilbert spectrum based on the sum of the IMFs  $c_1$  to  $c_3$ , to determine the time scale for the variability of Figure 16.

#### ACKNOWLEDGEMENTS

This research is supported in part by a grant from N00014-98-F-0412(NEH), and in part by the NASA RTOP project of the Ocean Processes Program (NEH and SRL), and in part by NOAA (NEH and SSPS). The encouragement by Dr Eric Lindstrom of NASA Headquarters, and Drs Lou Goodman and Thomas Curtin of ONR are deeply appreciated. WQ is supported by a grant to California Institute of Technology from NASA. SRL acknowledges the continuing support of Dr. Eric Lindstrom of NASA Headquarters. JEZ is supported by the Research Grants Council of Hong Kong. We would also like to express our special thanks to Professors O. M. Phillips of the Johns Hopkins University and T. Y. Wu of the California Institute of Technology, to whom the original paper on this method was dedicated.

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