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Minimum Error Estimates of Global Mean Temperature Through Optimal Arrangement of Gauges

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ABSTRACT

This paper considers the minimum mean squared error (MSE) incurred in estimating an idealized earth's global average temperature with a finite network of point gauges located over the globe. We use a spectral MSE formalism to find the optimal locations for N gauges in the problem of estimating the earth's global average temperature. Limiting MSE configurations are obtained as the limiting least error case for randomly distributed samples of size N. Our results suggest that for N greater than about 60, one can obtain estimates such that the amount of measured variance due to sampling error is less than 10%, a result likely to be acceptable to climatologists.

KEY WORDS: Global mean temperature; gauge location; Monte Carlo.

1. INTRODUCTION

There have been many studies on the estimation of the apparently increasing global average surface temperature (cf., Hansen and Lebedeff 1987; Jones *et al.* 1986a and 1986b; Houghton *et al.* 1990). Typically, these analyses consider the trend in the mean temperature and attempt to either forecast the time series or analyze the slope of a fitted model in order to

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investigate evidence of possible global warming. North, Shen, and Hardin (1992), denoted NSH hereafter, have presented a formalism useful in making quantitative assessments of the mean squared error (MSE) incurred in estimating the global average temperature with a finite number of distributed gauges over the sphere. This formalism is a spherical version of the planar case derived for area averaged rain rates (North and Nakamoto 1989). The technique is especially amenable to the development of least MSE strategies.

In the real world, the climatologist is constrained by the given locations of gauges comprising his gauge network. These strongly favor the inhabited and more developed parts of the world and in particular, the oceans are typically poorly represented. We will focus here on an idealized case of an earth which has rotationally invariant statistics on the sphere; i.e., the covariance of the surface temperature evaluated between a pair of points depends only on the great circle distance between the points. This tends to be the case for earth when temporal smoothing is very broad; i.e., the low frequency limit (Kim and North 1991).

We ask the very interesting theoretical question: For a given number of gauges, N, what is the least MSE one might obtain? Clearly, this will occur when the gauges are simultaneously 'equidistant' in some sense. The problem is identical to that of finding the equilibrium configuration of N equal point charges confined to a spherical surface. The problem of minimizing the potential energy is identical to the problem of finding the locations for the minimum MSE. Likewise, the problem is identical to the Gauss quadrature problem of finding the locations of N optimal points in performing an integral if the weights are constrained to be equal. There exist formulas for spherical quadrature in estimating the numerical solution for integrating a function on the surface of the sphere, but solutions do not exist for the optimal locations for a general N.

Many approaches could be taken to find the optimal locations of the N gauges. For example, one could take the partial derivatives of the MSE with respect to the latitude and longitude of the point gauges and set the results to zero. This would result in $2N$ nonlinear algebraic equations to be solved for $2N$ roots, a formidable problem for $N \sim 100$. A second approach is to view the problem as one in Newtonian mechanics and integrate the equations of motion of N particles confined to the spherical surface and under the influence of conservative repulsive forces. The introduction of friction would damp out oscillations. A third approach, and the one we adopt, is to introduce realizations of N gauges randomly located on the sphere. This Monte Carlo approach can be performed in various ways. The simplest approach is to perform straightforward random realizations looking for the configuration which provides the minimum MSE. Alternatives to this approach include attempting to determine if a given random configuration has no pair of gauges within a predetermined distance. However, this requires a considerable search for each realization. The brute force method will consider some configurations with gauges too close together, but increasing the number of iterations should overcome these cases and arrive at the desired result.

2. A SAMPLING ERROR FORMULA

Since the derivation of the MSE formalism has been shown in NSH, we present here only a very brief discussion of the problem. Identical assumptions presented in NSH on the smoothness of the autocorrelation function and the notational convenience of assuming a zero mean process are used here. We denote the long term temporal average surface temperature at a point $\hat{\boldsymbol{n}}$, a unit vector pointing from the earth's center as $\widetilde{\Theta}(\hat{\boldsymbol{n}})$. The assumption of spherical homogeneity leads to the form for the autocovariance function

$$\left\langle \widetilde{\Theta}(\hat{\boldsymbol{n}})\widetilde{\Theta}(\hat{\boldsymbol{n}}')\right\rangle = \sigma^2 \rho(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}') \tag{1}$$

where the angle brackets $\langle\,\rangle$ denote the ensemble average, $\sigma^2 = \langle\widetilde{\Theta}^2(\hat{\boldsymbol{n}})\rangle$ is the low frequency variance of the temperature field; and by definition $\rho(1) = 1$. As in NSH, it is convenient to develop $\rho(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}')$ into a Fourier Legendre series,

$$\rho(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}') = \sum_{l=0}^{\infty}(2l+1)\rho_l P_l(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}') \tag{2}$$

where equality holds under the assumption that the expansion converges almost everywhere.

Utilizing this approach, one can write the variance of global average temperature as

$$\sigma_{\oplus}^2 = \sigma^2\rho_0 \quad . \tag{3}$$

For a given array $\{N\}$ of gauges, NSH were able to derive the formula

$$\epsilon^2 = \frac{\sigma_{\oplus}^2}{\rho_0 N^2}\sum_{l=1}^{\infty}(2l+1)\rho_l \sum_{i,j=1}^{N} P_l(\hat{\boldsymbol{n}}_i\cdot\hat{\boldsymbol{n}}_j') \quad . \tag{4}$$

The ρ_l are the degree variance spectrum for the low frequency fluctuations of the temperature field. NSH also provide a parametric form for this spectrum under a simple stochastic model for the temperature field. Thus

$$\rho_l = \frac{\rho_0}{(\lambda_0^2\, l(l+1)+1)^2} \tag{5}$$

and, because of the normalization,

$$\sum_{l=0}^{\infty}(2l+1)\rho_l = 1 \tag{6}$$

which fixes ρ_0 for a given climate length scale λ_0. In what follows, the optimal configurations obviously do not depend on the specific value of λ_0 or, even to a large extent, upon the form of ρ_l (the only constraint is that the 'charges' should be repulsive!). The MSE, however, will depend sensitively on the value of λ_0 and the form of ρ_l, the climate length scale and the 'redness' of the spatial spectrum. The value of λ_0 is then to be $^{15}/_{60}$ (which leads to $\rho_0 = .0613$) to agree with data and the earlier assumptions. Finally, we choose a cutoff value L for the summation over the spherical harmonic indices imposing the assumption that the contributions to the overall variance from the spectral degrees higher than L are negligible.

As in NSH, we construct two figures of merit for a given network of gauges. The variance signal to noise

$$\Lambda_{\{N\}} \equiv \frac{\sigma_\oplus^2}{\epsilon^2} \tag{7}$$

and the percentage of measured variance due to sampling error,

$$V_{\{N\}} \equiv \frac{\epsilon^2}{\sigma_\oplus^2 + \epsilon^2} \tag{8}$$

$$= \frac{1}{1 + \Lambda_{\{N\}}} \quad . \tag{9}$$

3. Monte Carlo Evaluation of the Sampling Error

Natural questions arising from our formulation include:

1. At what rate does the sampling error tend to zero as the number of gauges increases to infinity?
2. What is the gain in adding gauges to the network in terms of sampling error reduction?
3. What role, if any, does the cutoff value L take in the calculation of the sampling error for small values of L; clearly it is equivalent to setting $\rho_l = 0$ for $l > L$ (band limited processes).

In the next few sections we present numerical estimates of the answers to the above questions by relating previous work and presenting the results of a Monte Carlo study of the sampling error. Works such as Hansen and Lebedeff (1987) pointed out that the spatial coverage of a single temperature gauge (calendar annual averages), owing to spatial correlation, may be as much as 1200 kilometers. Others have given numbers as high as 1500 kilometers for the 'reach' of the gauges, where reach is defined as the distance at which the correlation between two gauges falls to below $^1/_e$. If one takes a planet the size of earth, then the number of gauges needed to effectively 'cover' the planet is approximately 65. In looking for the minimum sampling error for a given number of gauges, N, how does one find the optimal locations for the gauges? This is analogous to equal interval sampling in time series analysis as what we seek is the most equipartitional configuration of N gauges, $\{N^*\}$. In unidimensional integration, the well-known Gaussian quadrature methods include techniques for estimating the error incurred in using the numerical technique. Gaussian quadrature also allows the user twice as many degrees of freedom in that the technique not only chooses the weights used at every point of evaluation, but also treats the locations themselves as random variables.

However, while one-dimensional domains are well established in the literature, spherical quadrature techniques (see, for example Stroud, 1971, or Stroud and Secrest 1966) essentially amount to finding the roots of many simultaneous equations, and formulas for calculating the errors and/or error rates do not exist. Thus, we took the Monte Carlo approach. The Monte Carlo approach is not expected to be a very efficient method for finding the precise locations of the N optimal gauges. However, we conjecture that the minimum is a rather broad one with small curvature (a frequency histogram of the MSE for random configurations should be skewed to the right); hence, our estimate of the minimum MSE should be rather robust.

The Monte Carlo approach to answer these questions consisted of a C program that generated random positions uniformly on the sphere. Given the number, N, of gauges, the program would randomly generate a configuration and then calculate the sampling error. At the end of the program, the minimum sampling error and the configuration corresponding to the minimum sampling error was output to a file. In order to eliminate identical solutions due to the inherent periodicity of random number generation algorithms, the program was executed in batches of 25,000 random configurations so that the random number seed could be changed. In the case of a cutoff value of $L = 15$, each network of gauges (sample sizes include $1, 2, \ldots, 10, 15, 20, \ldots, 100$) was considered in the Monte Carlo program a total of 100,000 times. That is, for each sample size N, 100,000 configurations were randomly simulated in lots of size 25,000 where each lot was initiated with a different random number seed. After this initial investigation, it was acknowledged that the cutoff value L might impose a length scale in the calculation and thus affect the conclusions. Therefore, we considered an additional 100,000 configurations for each sample size using cutoff values $L = 5$ and $L = 25$. In Table 1 we present the variance signal to noise index calculated for each sample size, and in Figure 1 we present a plot of these values for each cutoff value L. Future studies wishing to find the actual optimal configurations for each configuration might wish to use a more elegant search method, but the gain in signal to noise will be small. This can be seen by comparing the configurations given in Table 2 of NSH for only 2 gauges.

One can see from the results for the variance signal-to-noise ratio that the choice of cutoff may affect the results in size but that the relative dif-

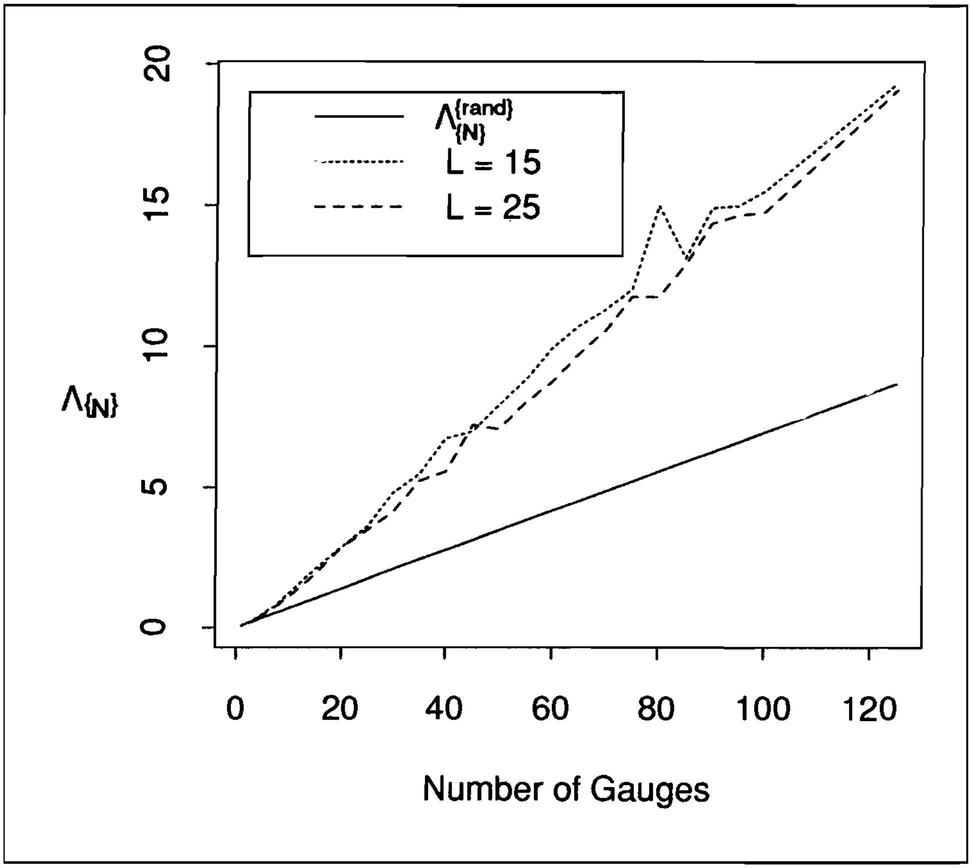

Figure 1. Signal to noise ratio of minimum Monte Carlo MSE gauge configuration for two cutoff values in the spherical harmonic expansion ($L = 15, 25$) and the theoretical mean sampling error for all possible configurations (shown as $\Lambda_{\{N\}}^{\{rand\}}$ for $L = 15$) versus the variance of an area average.

ference in the ratio between sample sizes is not affected. It is clear that a cutoff of $L = 5$ suppresses a significant amount of error variance and leads to unsatisfactory results for earth-like parameters ($\lambda_0 = {}^{15}/_{60}$). That the curves for $L = 15$ and $L = 25$ are so similar suggests that the results are reliable for these cutoff values. We expect a straight line for each of the plots since for very sparse designs ($N \ll 65$) the individual point estimates will be essentially independent, leading to the familiar standard error law. As N becomes sufficiently large, the straight line should turn over as the measurements become more dependent. In fact, we do find this for a sample size of 150, but it is not shown here since only 10,000 iterations of this sample size are performed due to excessive computer time and the fact that we are not looking for the exact point at which the line turns over. Since we had failed to capture so much of the variance between gauge locations using a cutoff

Number of Gauges	Λ_N $L = 5$	Λ_N $L = 15$	Λ_N $L = 25$
1	0.10	0.07	0.07
2	0.23	0.13	0.14
3	0.37	0.24	0.23
4	0.58	0.35	0.33
5	0.78	0.46	0.44
6	0.96	0.61	0.57
7	1.50	0.73	0.69
8	1.83	0.88	0.83
9	2.24	1.05	0.99
10	2.83	1.23	1.11
15	4.66	2.08	1.90
20	6.43	2.87	2.87
25	11.22	3.61	3.46
30	9.94	4.80	4.11
35	14.86	5.47	5.23
40	15.27	6.74	5.57
45	20.83	6.98	7.22
50	28.63	7.94	7.08
55	23.07	8.81	8.02
60	25.51	9.95	8.74
65	25.61	10.73	9.74
70	24.78	11.31	10.56
75	25.92	12.02	11.77
80	37.18	14.99	11.75
85	32.77	13.13	12.90
90	33.84	14.87	14.34
95	37.64	14.97	14.63
100	39.46	15.48	14.74
125	54.79	19.32	19.09

Table 1. Signal-to-noise ratio of sampling error to the variance of an area average for minimum Monte Carlo MSE gauge configurations for various cutoff values in the spherical harmonic expansion ($L = 5, 15, 25$).

value of $L = 5$, we do not include them in the figures, but show the values in the table so that comparisons may be made. Instead, we show the values associated with the overall mean of randomly placed gauges. In section 4 of NSH, the signal-to-noise ratio for averaging over all configurations of N gauges was shown to have the relationship

$$\Lambda_{\{N\}}^{rand} = N\Lambda_{\{1\}} \tag{10}$$

where the sampling error variance incurred on the average for N randomly located gauges is the same as for N independent measurements with a single gauge.

In Table 2, we present the percent contribution to the total variance by the sampling error and a plot of the values are given in Figure 2. One can see that to decrease the percent variance to below 10%, using a cutoff value of 15, necessitates a configuration of about 55 or 60 gauges. In the following section we compare our optimal values with those expected from a specific network of 63 gauges commonly used in climatology.

4. Angell-Korshover Configuration

In a study of tropospheric and stratospheric temperature variations, Angell and Korshover (1983) used a network of 63 radiosonde stations for data collection. This configuration has also been used for global surface temperature estimation. They characterized the network as "well-distributed" and in this section we would like to investigate this claim. Recall that these stations were used in a study of the real earth's climate while we have considered a planet with rotationally invariant statistics. Our aim in studying this paper is purely to investigate the available configuration of earth temperature gauges in our present analysis of sampling errors on our simplified planet. A study of the errors of the regional, zonal, and global means in terms of the trend and the standard deviation has been presented (Trenberth and Olson 1991) using simulations with numerical atmospheric models. We believe our approach is more general and leads to additional insight into the estimation problem. The locations of the 63 stations are shown in Figure 3.

If we use this network of gauges and insert the locations into (9), we obtain

$$V_{\mathbf{Angell}} = 14.89\% \ . \tag{11}$$

One can see that this compares with the optimal configurations of between 35 and 40 gauges. In fact, Figure 2 shows that the decrease in percent contribution to variance by adding gauges is very low once we have 40 gauges. This means that the cost associated with adding temperature gauges may outweigh any gain in precision which is what we see in this example.

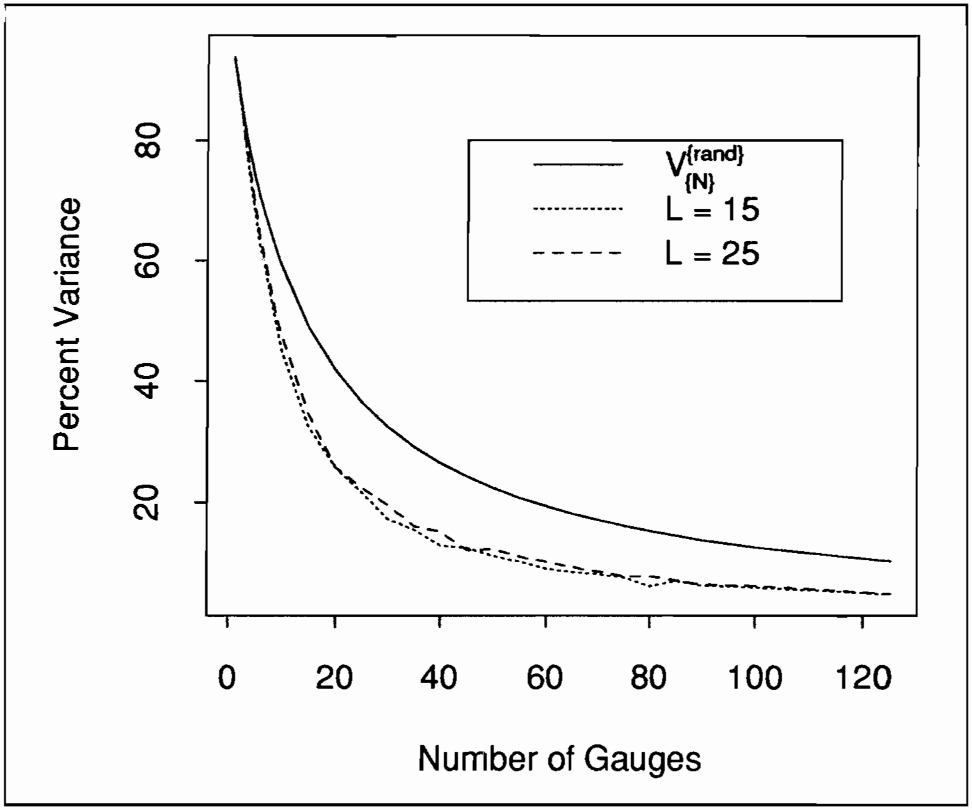

Figure 2. Percent of measured variance due to sampling error for the minimum Monte Carlo MSE gauge configurations for two cutoff values in the spherical harmonic expansion ($L = 15, 25$) and the theoretical mean value for all possible configurations (shown as $V^{\{rand\}}_{\{N\}}$ using $L = 15$) versus the number of gauges.

An interesting comparison with optimal locations can be made by examining the histograms provided in Figures 4 and 5. Figure 5 graphically depicts that the value obtained for the Angell-Korshover is at about the tenth percentile for random configurations of 60 gauges and Figure 4 shows the minimum for 40 gauges to be slightly less. The skewness exhibited in both of these histograms is not as pronounced as we had anticipated, yet still appears to be long tailed to the right. In fact, the minimum value for 60 gauges, as seen in Table 2, is 8.53%.

5. Conclusions

The addition of gauges to an optimally located network of 45 or 50 gauges has little effect on minimizing the MSE for our model earth. In terms of cost, there is little return in terms of relative MSE reduction for

Number of Gauges	V_N $L = 5$	V_N $L = 15$	V_N $L = 25$
1	91.23	93.51	93.74
2	81.56	88.41	87.47
3	72.80	80.56	81.27
4	63.38	74.18	75.15
5	56.09	68.31	69.45
6	50.91	62.13	63.67
7	40.01	57.74	59.02
8	35.32	53.21	54.59
9	30.82	48.73	50.29
10	26.12	44.81	47.46
15	17.65	32.44	34.54
20	13.46	25.82	25.82
25	8.18	21.71	22.41
30	9.14	17.25	19.56
35	6.30	15.45	16.05
40	6.15	12.93	15.23
45	4.58	12.53	12.17
50	3.38	11.18	12.38
55	4.15	10.20	11.09
60	3.77	9.14	10.26
65	3.76	8.53	9.32
70	3.88	8.13	8.65
75	3.71	7.68	7.83
80	2.62	6.25	7.84
85	2.96	7.08	7.19
90	2.87	6.30	6.52
95	2.59	6.26	6.40
100	2.47	6.07	6.35
125	1.79	4.92	4.98

Table 2. Percent of measured variance due to sampling error for the minimum Monte Carlo MSE gauge locations versus the number of gauges N for various cutoff values in the spherical harmonic expansion ($L = 5, 15, 25$).

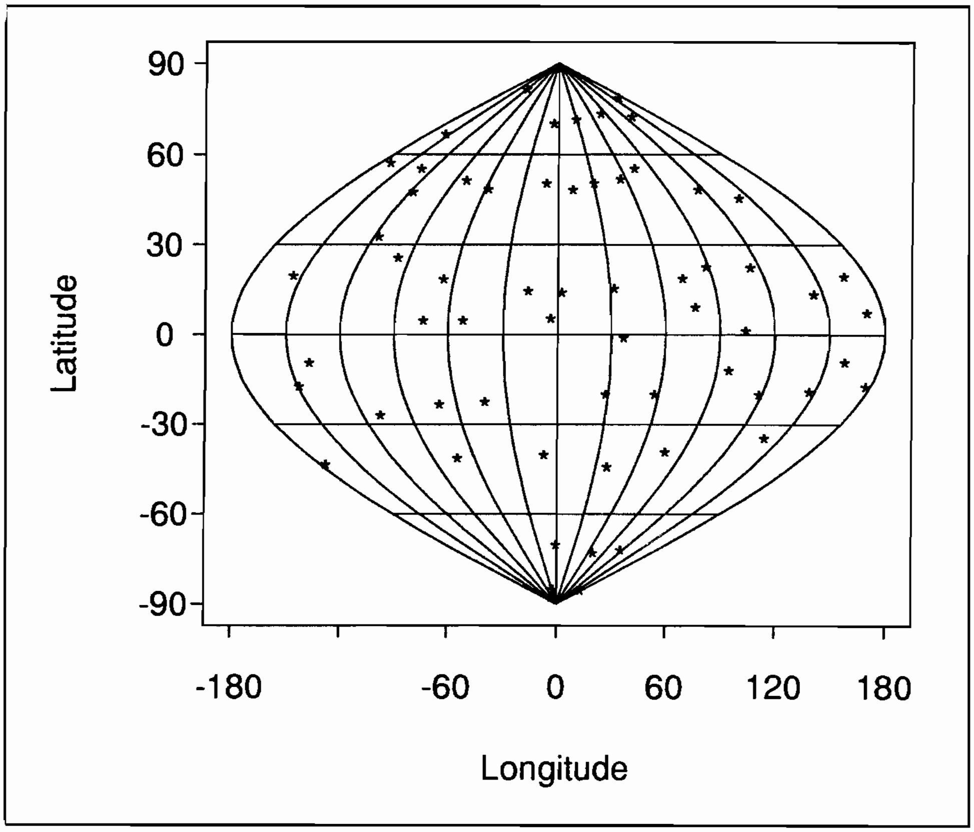

Figure 3. Sinusoidal projection of Angell/Korshover temperature gauge locations.

adding a small number of gauges. The preceding section also illustrates the gain in minimizing MSE by considering optimally located gauges. One could instead consider finding optimal weights for a given network of gauges instead of changing the design. This problem may be of more practical concern to climatologists as they do not have the freedom to choose gauge locations (though they do have the freedom to choose the best subset of locations from given data). Most studies would use all available data. Our emphasis on the best subset is in the sense of the relative gain in adding locations. Optimal weighting is similar to the optimal interpolation problems found in mining geostatistics examined with various kriging methods. Another generalization to the proposed approach to sampling error estimation we would like to consider is adding the time component to the analysis. This latter will break the rotational symmetry since ocean and land exhibit different correlation lengths for time smoothing of less than 5 years (see, for example, Kim and North 1991).

Our spectral MSE formalism leads to the optimal error versus number of gauges curve. This should serve as a useful guide to climatologists in

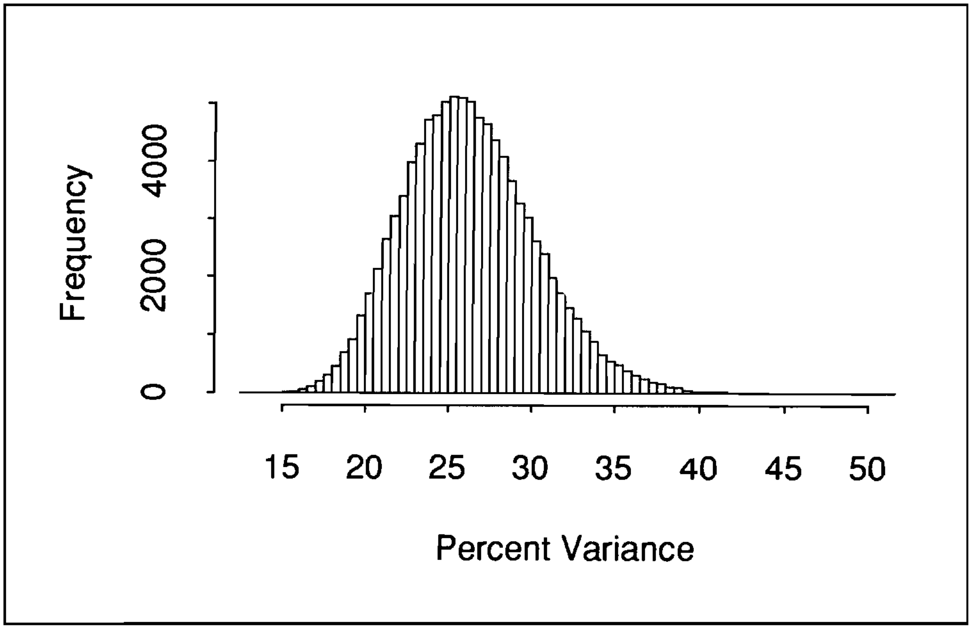

Figure 4. Frequency histogram of the percent variation explained by the network of 40 gauges for 100,000 random configurations using cutoff value $L = 15$ in the spherical harmonic expansion. The mean of these configurations is 26.26% while $1/(1+40\Lambda_{\{1\}}^{rand}) = 26.47\%$ showing agreement with the earlier theoretical results.

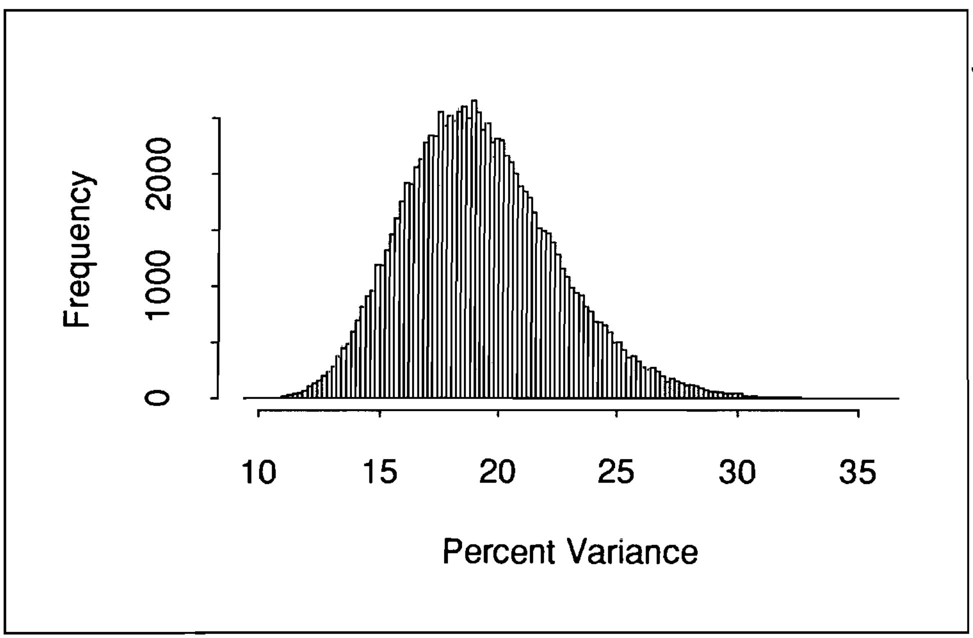

Figure 5. Frequency histogram of the percent variation explained by the network of 60 gauges for 100,000 random configurations using cutoff value $L = 15$ in the spherical harmonic expansion. The mean of these configurations is 19.24% while $1/(1+60\Lambda_{\{1\}}^{rand}) = 19.36\%$ showing agreement with the earlier theoretical results.

their design of global observing systems. We are intrigued by a class of other estimation problems in spherical geometry which are analogous to those on the real line encountered in equal interval sampling in time series analysis. For example, in a band limited process (one for which $\rho_l = 0$ for $l > L$), a finite configuration $N > N_0(L)$ could be used to calculate spectral quantities with zero error in analogy to the famous sampling theorem. Other interesting problems include assessment of aliasing in the estimation of the spherical harmonic degree spectrum.

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