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Tailing wavetrain generation in precursor soliton generation in single-layer flow *

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Abstract A theory of tailing wavetrain generation in the precursor soliton generation in single-layer flow is presented in terms of Whitham's averaged method in the present paper. The group characters of the tailing wavetrain generation are represented by the evolution equations of roots of a cubic algebraic equation resulting from the fKdV equation in the single-layer flow without source term. Based on the evolution equations, the group velocity of the tailing wavetrain is found theoretically, furthermore a theoretical solution of the tailing wavetrain generation is found in terms of the evolution equations. To examine the theoretical results, a numerical calculation of fKdV equation in single-layer flow is carried out in the laboratory frame, at the same time a comparison is also done. The comparison between theoretical and numerical results shows they are in good agreement.

Keywords: tailing wavetrain, precursor soliton, single-layer flow.

The precursor soliton generation is a kind of problem of nonlinear wave generation forced by local external sources in 2-D system. Since 1986, some authors have devoted themselves to study the wave factors in the precursor soliton generation: the generating amplitude, period and velocities, etc.[1—11]. In previous studies, the reasonable solution of the tailing wavetrain generation in the precursor soliton generation could not be given. Smyth[2] studied the tailing wavetrain generation in the stratified flow over topography near resonance, and gave some solutions of the tailing wavetrain generation with truncated moduli m_s and m_w of the modulus m of the complete elliptic integral of the first kind. The solutions imply that Whitham's averaged KdV equation cannot be used to describe the tailing wavetrain generation completely or the wave component of the tailing wavetrains can only focus on the high end of the modulus m. To obtain a reasonable solution of the tailing wavetrian generation in single-layer flow, a predictable theory taking into consideration all the modulus m is studied in this paper.

1 Stationary solution of KdV equation in single-layer flow

It is known that the solution of the tailing wavetrain generation in the precursor soliton

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generation satisfies the fKdV equation, however when generating time is long enough, the tailing wavetrain is far from topography (or forced source), the influence of the topography on the tailing wavetrain generation can be ignored. Therefore the tailing wavetrain can be considered as a free wavetrain and the fKdV equation without source term can be used to study the tailing wavetrain generation. The fKdV equation in the single-layer flow without source term is as follows[4,5]:

$$\eta_{t'} + (F-1)\eta_{x'} + m\eta\eta_{x'} + n\eta_{x'x'x'} = 0, \tag{1}$$

where F is moving velocity of the topography, x' and t' are the horizontal coordinate and generating time in the laboratory frame respectively, η is the wave elevation relative to the level in the rest. Introduce transform

$$u = mn^{-1/3}[\eta + (F-1)/m]/6, \qquad x = n^{-1/3}x', \quad t = t', \tag{2}$$

and substitute eq. (2) into eq. (1), yields

$$u_t + 6uu_x + u_{xxx} = 0. \tag{3}$$

It is shown that there exists the stationary wave solution in eq. (3). Introducing a phase X, namely

$$X = x - Vt, \qquad u = u(X), \tag{4}$$

where V is the velocity of phase, and using (4), we have a related equation to eq. (3):

$$F(u) = u_X^2/2 = -u^3 + Vu^2/2 + Bu - A, \tag{5}$$

where A and B are the integral constants. From the eq.(5), it is follows that

$$u = \beta \pm 2a\mathrm{cn}^2[\sqrt{a/s^2}(x - Vt), s^2], \tag{6}$$

where a is the amplitude of the stationary waves and is defined as

$$a = (\beta - \alpha)/2, \tag{7}$$

and s^2 is the modulus of the elliptic integral, namely,

$$0 \leqslant s^2 = (\beta - \alpha)/(\gamma - \alpha) \leqslant 1, \quad \alpha \leqslant \beta \leqslant \gamma, \tag{8}$$

where α, β and γ are the roots of the related equation. It can be proven that the period of the stationary cnoidal waves in (6) is $2K(s^2)$, where $K(s^2)$ is the complete elliptic integral of the first kind. By analogy to linear waves, the wavelength, wavenumber and velocity of the phase of the waves are, respectively,

$$L = 2sK(s^2)/\sqrt{a}, \quad \chi = \pi\sqrt{a}/[sK(s^2)], \quad V = 2[3\beta + 2a(2s^2 - 1)/s^2]. \tag{9}$$

Formula (6) gives a form of the solution of the tailing wavetrain generation in the precursor soliton generation in the single-layer flow.

2 Averaged KdV equation

An averaged KdV equation is derived under condition of $\alpha \leqslant \beta \leqslant \gamma$ in this section[12] as follows:

$$\frac{\partial(\alpha+\gamma)}{\partial t}+\left[V+2\Omega_A\frac{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}{\Omega_\gamma-\Omega_\alpha}\right]\frac{\partial(\alpha+\gamma)}{\partial x}=0, \tag{10}$$

where

$$\Omega(V,A,B)=-\sqrt{2}\oint\sqrt{F[u(X)]}\mathrm{d}u \tag{11}$$

and

$$\Omega_A=\chi^{-1},\quad \Omega_\alpha=\sqrt{2}\frac{(\beta-\alpha)^2}{\sqrt{(\gamma-\alpha)}}\left[\frac{(1+s^2)E(s^2)-K(s^2)}{s^4}+A_4\right],$$

$$\Omega_\gamma=-\sqrt{2}\frac{(\beta-\alpha)^2}{\sqrt{(\gamma-\alpha)}}\left[\frac{K(s^2)-E(s^2)}{s^2}-A_4\right]. \tag{12}$$

In (12), $E(s^2)$ is the complete elliptic integral of the second kind and A_4 is Jacobian elliptic integral with even power 4.

Substituting eq. (12) into eq. (11), yields

$$\frac{\partial\beta}{\partial t}+\left[V-\frac{4as'^2K(s^2)}{E(s^2)-s'^2K(s^2)}\right]\frac{\partial\beta}{\partial x}=0, \tag{13}$$

where s'^2 is the complementary modulus of s^2. Hence, along characteristic

$$\frac{\mathrm{d}x}{\mathrm{d}t}=C_g=V-\frac{4as'^2K(s^2)}{E(s^2)-s'^2K(s^2)}, \tag{14}$$

eq.(13) can be rewritten as

$$\frac{\partial\beta}{\partial t}+C_g\frac{\partial\beta}{\partial x}=0. \tag{15}$$

Eq.(14) gives the group velocity of the dispersion wavetrains and eq.(15) describes the evolution of the wavetrains.

3 Tailing wavetrain generation in single-layer flow

According to the numerical results of the precursor soliton generation, the wave shape of the tailing wavetrains possess a self-similar character, i.e. $\mathrm{d}x/\mathrm{d}t \Rightarrow x/t$. To determine the phase of the tailing wavetrain generation, an averaged value of u can be derived by our definition as follows:

$$\overline{u}=\alpha-2aD(s^2)/K(s^2), \tag{16}$$

where $D(s^2)$ is the standard notation of the complete elliptic integral[12]. At the first zero-crossings

$x = x_{\text{down}}$ and when $k \to 1$ ($\beta = \gamma$), the waves at the first zero-crossings approach to the solitary waves and (16) gives the exact value of $\bar{u}$ for the solitary waves. However, the waves at the first zero-crossings are conoidal waves, but not the solitary waves. Hence, for the tailing wavetrain generation, the averaged value $\bar{u}$ cannot be used to find the solution of the tailing wavetrain generation. To determine the solution, we give the following physical consideration, that is

$$\bar{u} = (\alpha + \gamma)/2 = \alpha + (\gamma - \alpha)/2. \tag{17}$$

As a reasonable condition of the tailing wavetrain generation, the amplitude of the tailing wavetrain at the first zero-crossing should be equal to twice the mean water level in the depressed water region in the frame (x, t), i.e. $2H_+$. Let

$$\alpha = h, \tag{18}$$

where h is a distance from the surface of the depressed water region at the end of $s = 1$ to the bottom of tank in the frame (x, t). Thus eq. (17) is converted in to

$$\bar{u} = h + H_+. \tag{19}$$

From eq. (17), it follows that

$$\gamma = h + 2H_+, \tag{20}$$

and by eq. (8), it follows that

$$\beta = h + 2s^2 H_+. \tag{21}$$

Substituting eqs. (18) and (21) into eq. (7) yields

$$a = s^2 H_+. \tag{22}$$

Thus, the wave factors described in eq. (9) become

$$L = 2sK(s^2)/\sqrt{H_+},\ \chi = \pi\sqrt{H_+}/[sK(s^2)],\ V = 2[3h + 2H_+(1 + s^2)]. \tag{23}$$

As $s = 1$, the velocity of the phase of the stationary wave

$$V_0 = 6h + 8H_+ \tag{24}$$

can be removed, thus the velocity of the wave group is obtained as

$$C_g = -4H_+\{(1 - s^2) + s^2 s'^2 K(s^2)/[E(s^2) - s'^2 K(s^2)]\}. \tag{25}$$

Integrating eq. (14), yields

$$x = C_g t + C, \tag{26}$$

where C is an integral constant. It is expected that as $s = 1$, $x = x_{\text{down}}$, thus $C = x_{\text{down}} = U_+ t$, where U_+ is moving velocity of the first zero-crossing in the frame (x, t). Hence, it follows that

$$x = (U_+ + C_g)t. \tag{27}$$

Eq. (14) gives two selections on the phases. In order to obtain a wavetrain moved to right, a reasonable selection is

$$\Theta = -[x - (U_+ + C_g)t]. \tag{28}$$

Upon substituting eq. (28) into eq. (6), it follows that

$$u = h + 2s^2 H_+ + 2s^2 H_+ \operatorname{cn}^2(\sqrt{H_+}\Theta). \tag{29}$$

It is shown that eq. (29) is a solution for the dispersive wavetrain, but it does not satisfy the generating conditions of the tailing wavetrains. At the first zero-crossings ($s = 1$), eq. (29) gives

$$u = h + 4H_+. \tag{30}$$

From eq. (20), it is known that the wave elevation $u = \gamma$ at the end of $s = 1$, so the term $2s^2 H_+$ should be removed from eq. (29). Thus it follows that

$$u = h + 2s^2 H_+ \operatorname{cn}^2(\sqrt{H_+}\Theta). \tag{31}$$

At the end of $s = 0$, eq. (19) must be satisfied so that the mean level is equal to the level in the rest. So at the end of $s = 0$, the H_+ should be added to eq. (31). From $s = 1$ to $s = 0$, a resulting contribution is $(1 - s^2)H_+$. Finally, an intelligent solution is given as

$$u = h + 2s^2 H_+ \operatorname{cn}^2(\sqrt{H_+}\Theta) + (1 - s^2)H_+. \tag{32}$$

In order to examine the theoretical results (28) and (32), both must be converted to the laboratory frame (x', t'). From (2), it follows that

$$U_+ = n^{-1/3} U'_+. \tag{33}$$

Because $H_+ > 0$ and $H'_+ < 0$, a scale transform of the H_+ is

$$H_+ = -(m/6) n^{-1/3} H'_+, \tag{34}$$

and

$$h = (m/6) n^{-1/3} h' = (m/6) n^{-1/3}[1 + H'_+ + (F - 1)/m], \tag{35}$$

where h' is a distance from surface of the depressed water region to the bottom of the tank, and U'_+ and H'_+ are the velocity of the first zero-crossing and the level in the depressed water region in the laboratory frame (x', t') respectively, which were found by Xu et al.[5,7,8,10] theoretically as follows:

$$U'_+ = -m[\mathscr{H} - (F-1)/m]/2, \tag{36}$$

$$H'_+ = -\mathscr{H} - (F-1)/m. \tag{37}$$

Substituting eqs. (32)—(35) into (2) yields

$$\eta = 1 + H'_+ + 2s^2 H'_+ \operatorname{cn}^2\{[\sqrt{-(m/6)n^{-1/3}H'_+}\Theta'],\ s^2\} - (1-s^2)H'_+,\ s^2 = 0,\ 1, \tag{38}$$

where

$$\Theta' = -n^{-1/3}\{x' - [U'_+ + 4H'_+(m/6)\Omega']t'\}, \tag{39}$$

$$\Omega' = (1-s^2) + s^2 s'^2 K(s^2)/[E(s^2) - s'^2 K(s^2)],\quad s^2 = 0,1. \tag{40}$$

Eqs. (38)—(40) are the solution of the tailing wavetrain generation in the precursor soliton generation in the single-layer flow in the laboratory frame with the parameters $m = -3/2$ and $n = -1/6$.

4 Comparison and conclusion

A numerical calculation of the fKdV equation in the single-layer flow by using the difference method[13] is carried out to examine the theory in Section 3. The fKdV equation in the absolute frame is

$$\eta_{t'} - (1+m\eta)\eta_{x'} + n\eta_{x'x'x'} = f(x' + Ft')_{x'}/2, \tag{41}$$

where $m = -3/2$ and $n = -1/6$. The semicircular topography function is

$$f(x' + Ft') = \sqrt{r^2 - (x' + Ft')^2}, \tag{42}$$

and the boundary and initial conditions are

$$\eta(\pm\infty) = \eta_{x'}(\pm\infty) = \eta_{x'x'}(\pm\infty) = \eta(x',0) = 0. \tag{43}$$

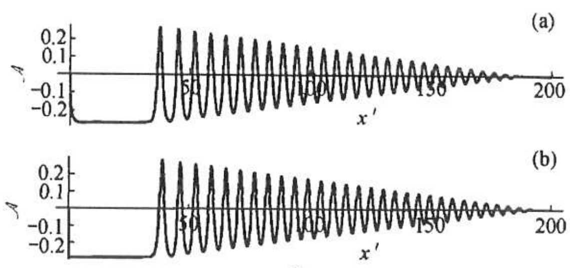

Fig. 1. Numerical (a) and theoretical (b) results of tailing wavetrain generation

A numerical result with a radius of the semicircular topography $r = 0.230\,8$ at the resonant point is shown in fig. 1 (a), in which $\mathscr{A}$ is the amplitude of the tailing wavetrain. From the numerical results, it is shown that the generating period of the precursor solitons and the level of the depressed water region at the resonant point are $\tau_0 = 31.5013$ and $H'_+(F=1) = -\mathscr{H} = -0.265\,95$ respectively. In fig. 1 (a), it is also shown that as there are 5.8 solitons in the upstream of the topography, the tailing wavetrain includes 28 wave crests. By ref. [5], the theoretical generating period of the precursor solitons and the level in the depressed water region at the resonant point are $\tau_0 = 28.894\,3$ and $H'_+(F=1) = -\mathscr{H} = -0.283\,209\,5$ respectively. Substituting the theoretical results into eqs. (39) and (40), we have the total coefficient of the phase of the conoidal wave 30.932 87. Using eq. (38), we calculated theoretically the tailing wavetrain generation under the condition of 5.8 precursor solitons in the upstream of the topography. The theoretical results is shown in fig. 1 (b) with parameters $m = -3/2$, $n = -1/6$, $p = 0.087\,09$ and $t' = t = 5.8\tau_0 = 167.586\,94$. From the fig. 1(b), it is known that there are 29.5 wave crests in the tailing wavetrain. From the above results, it is known that the theoretical and numerical results are in good agreement. It should be pointed out that the accuracy of the numerical and theoretical calculation is restricted by the truncation of the difference and the asymptotic approximation, and a departure between the numerical and theoretical result at the resonant points for this example is 5.084 7%. Finally, it must be pointed out that the theory in this paper is a predictable theory of the tailing wavetrain generation in the problem of the precursor soliton generation in single-layer flow as long as the ambient parameters such as the depth of water, density of fluid, topography, velocity of topography are given.

From the theoretical eqs. (38)—(40), it is shown that as the moving velocity of the topography is at the resonant points, there exists an invariable in the problem of the tailing wavetrain generation in single-layer flow, i.e. the ratio of the width of the generating region of the tailing wavetrain to that of the depressed water region is 4. This conclusion will be examined in the future.

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