Eur. J. Mech., B/Fluids, **9**, n° 1, 59-74, 1990

A new equilibrium of subcritical flow over an obstruction in a channel of arbitrary cross section

S. S. P. SHEN * and M. C. SHEN **

ABSTRACT. — In this paper a new steady state solution of a current of ideal fluid over an obstruction in a channel of arbitrary cross section is found. This new solution is a hydraulic fall. The existence of such a solution is demonstrated by studying a forced Korteweg-de Vries equation. It turns out this solution appears only when the upstream subcritical current velocity u_0 equals to a special value u_L. Numerical solutions are presented.

1. Introduction

Since the pioneering experimental work of Huang *et al.* [1982] and the remarkable numerical findings of Wu and Wu [1982], and Forbes & Schwartz [1982], lately there has been growing interest in studying the surface and internal waves of a fluid flow over an obstruction. This is not only because of the theoretical interests of the problem, but also because of its important applications in many fields, such as meteorology [1987] and oceanography [1987].

In the case of a two dimensional channel and ideal fluid, most features of the free surface profiles, which depend on the velocities of the upstream current, have been known. When the upstream current is subcritical ($F<1$), Fobres & Schwartz [1982] numerically found that the free surface consists of a downstream cnoidal wave matched with an upstream null solution. Here F is the upstream Froude number, which is the ratio of the velocity of the uniform upstream current to the phase velocity os shallow water waves. They obtained the solution by solving the exact nonlinear boundary value problem of a Laplace equation. Their numerical output fails to converge when the Froude number F is too close to unity, at which the upstream flow is called critical. When F is very close to unity either from positive side or negative side, at which the upstream flow is called transcritical, Wu & Wu [1982] conjectured that there are no steady-state solutions. By solving shallow-water equations of Boussinesq type numerically, they found that there are wave solitons periodically generated at the obstruction and

* Department of Mathematics and Climate System Research Program, College of Geoscience, Texas A & M University, College Station, TX 77843, USA.

* Department of Mathematics, University of Wisconsin, Madison, WI 53706, U.S.A.

0997-7546/90/01 59 16/$ 3.60/

radiated upstream. Such a soliton radiation phenomenon has also been numerically discovered by solving forced shallow-water equation of Korteweg-de Vries type ([Akylas, 1984], [Cole, 1985], [Mei, 1986], [Grismshaw & Smyth 1986] and [Wu, 1987]). Ertekin *et al.* [1986] numerically found the similar phenomenon in the three dimensional situation by the Green-Naghdi theory. When the upstream flow is supercritical ($F>1$), using a method similar to that in [F. & S., 1982], Vanden Broeck [1987] numerically obtained two stationary solitory wave solutions if F is sufficiently large.

Recently Forbes [1988] numerically and experimentally showed that a hydraulic fall may occur when an ideal fluid flow passes an obstruction at the bottom of a two dimensional channel. This water fall is not included in the classes of solutions aforementioned, and happens only when the upstream Froude number takes a special value. The transition region of the cascade of the free surface the upstream large depth to the downstream smaller depth is over the obstruction. This phenomenon was theoretically confirmed by Shen *et al.* [1989] via asymptotic analysis. Under the long wave assumpation, they were able to asymptotically reduce the Euler equations and the boundary conditions to a forced Korteweg-de Vries (fK-dV) equation for η_1. There $H(1+\varepsilon\eta_1(x))$ is the profile of the free surface; H is the upstream depth; $\varepsilon=(H/L)^2 \ll 1$ with L being the horizontal scale. x is the longitudinal coordinate of the channel. We write $F=1+\varepsilon\lambda$. When λ is sufficiently negative, the free surface consists of a null solution upstream matched with a cnoidal wave downstream. The period T of the downstream cnoidal wave increases as λ increases, and eventually $T=\infty$ as $\lambda=\lambda_L<0$. Hence the downstream becomes wave free as $\lambda=\lambda_L$. This is the hydraulic fall [F., 1988].

A hydraulic fall is a different phenomenon from the well known hydraulic jump, which is a transition from the upstream surface level to a higher downstream surface level. A hydraulic jump occurs in a rectangular channel of flat bottom only when the upstream flow is supercritical [Yih, 1979]. Some mechanical energy is transformed to the internal energy in the jump region, where the water is turbulent. In constrast to the hydraulic jump, a hydraulic fall is a transition from the upstream surface level to a lower downstream surface level. A hydraulic fall occuts only when there is an external forcing (such as a bump on the bottom) and the velocity of the subcritical upstream flow is equal to a certain value. The entire flow field is laminar and there is no loss of mechanical energy in the fall region.

The present paper is addressed to the same hydralic fall phenomenon. However we are studying the flow in a domain of much more complicated geometry: a channel of arbitrary cross section (see *Fig.* 1). It appears that water fall solution in such a domain is new. It is well known that long surface waves on water in a three dimensional channel is asymptotically two dimensional ([Peters, 1966], [Stoker, 1957], and [Shen, 1975]). Namely the profile of the free surface is independent of the spanwise coordinate. It is this property of the flow that stimulates us to write this paper. The small obstruction may be installed on the wall of the channel or may equivalently consist of a distributed pressure on the free surface. When the upstream current u_0 is near its critical state u_c, *i.e.*, $|u_0-u_c|=0(\varepsilon)$, the profile of the free surface asymptotically yields a forced Korteweg-de Vries (fK-dV) equation. Here H and L are the transverse and longitudinal length scales respectively, and $\varepsilon=(H/L)^2 \ll 1$.

As the upstream subcritical velocity u_0 is sufficiently small, the fK-dV possesses a solution. This solution vanishes upstream and consists of cnoidal waves downstream. The period T of the downstream cnoidal waves increases as the upstream subcritical current velocity u_0 increases. The period $T \to \infty$ as u_0 approaches a limit value $u_L < u_c$. Thus a hydraulic fall forms in a channel of arbitrary cross section.

The objective of this paper is to show the existence of the above limit u_L, and hence to confirm the existence of the hydraulic fall in a channel of arbitrary cross section. Apparently this u_L depends on the size of the obstruction. In Sect. 2, we outline the derivation of a fK-dV equation. The details are given in the appendix. In Sect. 3, complete analysis on the subcritical solution of the fK-dV equation is given. The existence of u_L is shown in this section. In Sect. 4, some typical profiles of the subcritical flow and the relation-ship between u_L and the size of the obstruction are numerically computed. Results are presented for the cases of rectangular and triangular channels.

2. A stationary forced Korteweg-de Vries equation

In this section, we derive the differential equation for the elevation of the free surface of a stationary current over an obstruction in a channel. The shallow water assumption is used. If H and L are transverse and longitudinal length scales respectively, then H may be the typical depth of upstream and L may be the typical wave length. The elevation η^* of the free surface is assumed to be of order εH. The shallow water assumpation is that $\varepsilon H/L = \varepsilon^{3/2} \ll 1$.

Let x^*-axis be aligned along the longitudinal direction of the channel, the y^*-axis along the spanwise direction and z^*-axis vertically in the opposite direction to gravitation. The unperturbed free surface is taken as the x^*, y^* plane. The equation of the boundary of the channel is $h^*(x^*, y^*, z^*)=0$. The equation of free surface is $z^* = \eta^*(x^*, y^*)$. In this entire section, the superscript* denotes dimensional quantities. The equations of motion and boundary conditions of a steady state flow are

(1) $$u^*_{x^*} + v^*_{y^*} + w^*_{z^*} = 0,$$

(2) $$u^* u^*_{x^*} + v^* u^*_{y^*} + w^* u^*_{z^*} = -p^*_{x^*}/\rho^*,$$

(3) $$u^* v^*_{x^*} + v^* v^*_{y^*} + w^* v^*_{z^*} = -p^*_{y^*}/\rho^*,$$

(4) $$u^* w^*_{x^*} + v^* w^*_{y^*} + w^* w^*_{z^*} = -g - p^*_{z^*}/\rho^*;$$

on the free surface $z^* = \eta^*(x^*, y^*)$,

(5) $$u^* \eta^*_{x^*} + v^* \eta^*_{y^*} - w^* = 0,$$

(6) $$p^* = \bar{p}^*(x^*);$$

on the wall of the channel $h^*(x^*, y^*, z^*)=0$,

(7) $$u^* h^*_{x^*} + v^* h^*_{y^*} + w^* h^*_{z^*} = 0.$$

Here (u^*, v^*, w^*) is velocity, ρ^* is density; p^* is pressure, g is the gravitational acceleration constant; and $\bar{p}^*$, which, assumed to be function of only x^*, is the distributed pressure on the free surface (see *Fig.* 1).

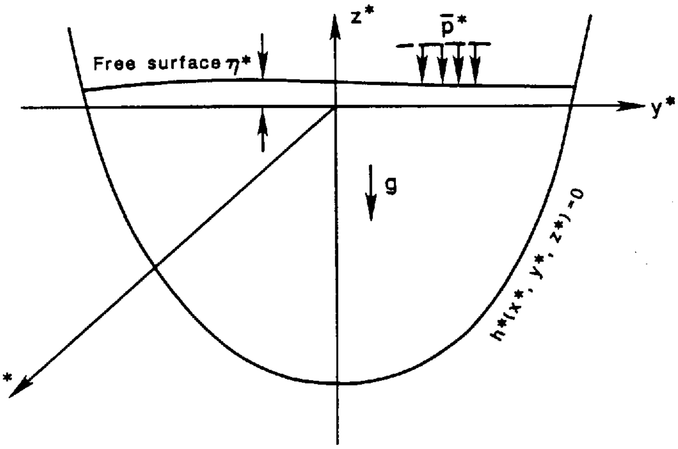

Fig. 1. — Configuration of the fluid domain: An ideal flid flow through a channel of arbitrary cross section with the free surface of the flow disturbed by a distributed pressure $\bar{p}^*(x^*)$.

We introduce the following dimensionless quantities to non-dimensionalize the equations (1) to (7).

$$(x, y, z) = (1/\mathrm{H})(\varepsilon^{1/2} x^*, y^*, z^*),$$

$$\eta = \eta^*/\mathrm{H}, \qquad p = p^*/(\rho^* g \mathrm{H}), \qquad \bar{p} = \varepsilon^2 \bar{p}^*/(\rho^* g \mathrm{H}) \quad \text{(small disturbance)},$$

$$(u, v, w) = (1/\sqrt{g\mathrm{H}})(u^*, \varepsilon^{1/2} w^*),$$

$$h_1 = \varepsilon^{-5/2} h^*_{x^*} \quad \text{(small obstruction)}, \qquad h_3 = h^*_{y^*}, \qquad h_3 = h^*_{z^*}.$$

The upstream uniform velocity u_0 is assumed to be near its critical state u_c. The exact expression is $u_0 = u_c + \varepsilon\lambda + O(\varepsilon^2)$. The unknowns (u, v, w, η, p) are assumed to be of the following assymptotic expansion:

$$(8) \quad (u, v, w, \eta, p) = (u_c, 0, 0, 0, -z) + \varepsilon(u_1 + \lambda, v_1, w_1, \eta_1, p_1) + \varepsilon^2(u_2, v_2, w_2, \eta_2, p_2) + O(\varepsilon^2).$$

Substituting (8) into the nondimensionalized equations of (1) to (7) and arranging the resulting equations according to the powers of ε, one can obtain two boundary value problems of order ε and ε^2 respectively. The solvability condition of the problem of order ε determines the critical velocity u_c and the one of the problem of order ε^2 yields the govering equation for η_1 which is a *f* K-*d*V equation.

Precisely, the solvability condition of the boundary value problem of order ε is

$$(\mathrm{A}/u_c^2 - b)\,\eta_{1x} = 0$$

where b and A are respectively the width and the area of the wet cross section of the channel. For nontrivial solutions, $\eta_{1x} \not\equiv 0$. This results in the dispersion relation

$$A = u_c^2 b. \tag{9}$$

Thus the critical velocity u_c is totally determined by the geometry of the channel and is given by

$$u_c = \sqrt{A/b}. \tag{10}$$

For a rectangular channel, $u_c = 1$. This is what was expected.

The solvability condition of the boundary value problem of order ε^2 is a stationary f K-dV equation:

$$\lambda m_1 \eta_{1x} + m_2 \eta_1 \eta_{1x} + m_3 \eta_{1xxx} = -f(x) \tag{11}$$

where

$$m_1 = -2b \sqrt{\frac{b}{A}}, \tag{12}$$

$$m_2 = 3\frac{b^2}{A} - \frac{b}{A}\int_\Gamma \varphi_{yy}\, dy, \tag{13}$$

$$m_3 = \frac{b}{A}\iint_D |\nabla \varphi|^2\, dy\, dz, \tag{14}$$

$$f(x) = b\bar{p}_x + h_1(x)\int_C \frac{ds}{\sqrt{h_1^2 + h_2^2}}. \tag{15}$$

In (12)-(15), for the integration domain D, the integration contours Γ and C, *see* Figure 2. $\nabla = (\partial/\partial y, \partial/\partial z)$, and $\varphi = \varphi(y, z)$ is a solution of the following Neumann problem:

$$\nabla^2 \varphi = 1 \quad \text{in D}, \tag{16}$$

$$\varphi_z = u_c^2 \quad \text{on } \Gamma, \tag{17}$$

$$\varphi_b = 0 \quad \text{on C} \tag{18}$$

where $\nabla^2 = (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$ and φ_n is the unit outward normal derivative of φ on C. The details of the derivations of (9) and (11) are given in the appendix of this paper.

From (12)-(15) and (16)-(18), we see that the coefficients m_1, m_2 and m_3 are entirely determined by the geometry of the channel. The velocity of the upstream current enters the equation (11) only by λ. It is this λ that controls the behavior of the solutions.

If D is a rectangle or a triangle, (16)-(18) can be solved very easily.

(i) If D is a rectangle of width b and depth d, then $\varphi_z = z + d$, $\varphi_y = 0$, $m_1 = -2b/\sqrt{d}$, $m_2 = 3\,\mathrm{b/d}$, $m_3 = \mathrm{bd}^2/3$, $f(x) = (\bar{p}_x(x) + h_1(x))\,b$.

(ii) If D is a triangle of width b and depth d, then $\varphi_y = y/2$, $\varphi_z = (z + d)/2$, $m_1 = -2\sqrt{2}\,b/\sqrt{d}$, $m_2 = 5\,b/d$, $m_3 = (12\,d^2 + b^2)\,b/(48\,d)$.

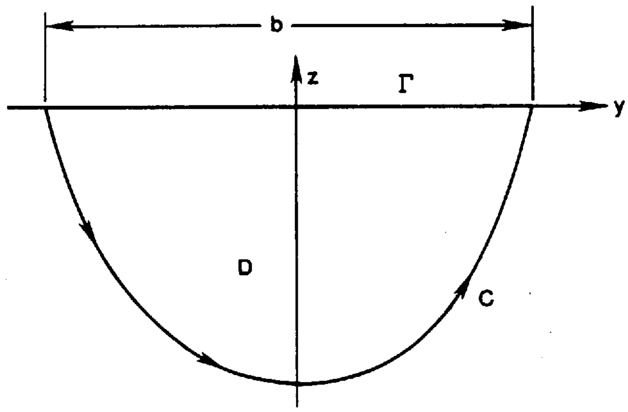

Fig. 2. – D is the wet cross section of the channel and D is also the integration domain of (14). $C \cup \Gamma = \partial D$ and C and Γ are integration contours of (13) and (15).

3. Existence of the new equilibrium: hydraulic fall

In this section, we will consider only subcritical flow (corresponding to $\lambda<0$) and show the existence of such a hydraulic fall solution, which happens when λ takes a special value $\lambda_L<0$.

By (12)-(14), we have $m_1<0$, $m_2>0$, $m_3>0$. (11) can be written as

$$\lambda\eta_{1x}+2\alpha\eta_1\eta_{1x}+\beta\eta_{1xxx}=r'(x) \tag{19}$$

where $\lambda<0$, $\alpha=m_2/(2m_1)<0$, $\beta=m_3/m_1<0$, $r'(x)=-f(x)/m_1$. If $\eta_1(x)$ is a solution of (19) and vanishes at $x=-\infty$ *together with its first derivative, then, from* $r(-\infty)=0$, η_1 satisfies

$$\lambda\eta_1+\alpha\eta_1^2+\beta\eta_{1xx}=r(x), \tag{20}$$

$$\eta_1(-\infty)=0,\ \eta_{1x}(-\infty)=0. \tag{21}$$

The right hand side of (20) is due to the obstruction on the boundary of the channel or due to the distributed pressure on the free surface. For pratical applications and experimental designs, it is often that $r(x)\geqq 0$, and r is continuous and nonvanishing in a bounded closed interval. Let $x_-=\inf\operatorname{supp}(r)$, $x_+=\sup\ \operatorname{supp}\ (r)$. Thus if $\eta_1(x)$ is a solution of (20)-(21), then $\eta_1(x)\equiv 0$ as $x\leqq x_-$.

Define a complete metric space

$$\mathscr{M}=\{u\,|\,u\in C([x_-,x_+]),\ \|u\|=\max_{x\leqq x\leqq x_+}|u(x)|\leqq M \text{ for some given positive constant } M\}.$$

For $x_-\leqq x\leqq x_+$, (20)-(21) can be converted into the integral equation

$$\eta_1(x)=\frac{1}{\sqrt{\beta\lambda}}\int_{x-}^{x}\sin\sqrt{\frac{\lambda}{\beta}}(x-\tau)\,[r(\tau)-\alpha\eta_1^2(\tau)]\,d\tau\equiv T(\eta_1). \tag{22}$$

It can readily be shown that if λ satisfies

$$(\alpha M/\sqrt{\beta\lambda})(x_+ - x_-) < 1/2, \tag{23}$$

$$(1/\sqrt{\beta\lambda})(\|r\|/M - \alpha M)(x_+ - x_-) \leqq 1, \tag{24}$$

then the map T defined by (22) is a contraction map in $\mathscr{M}$. Hence $T(\eta_1) = \eta_1$ has a unique solution in $\mathscr{M}$ by the contraction mapping theorem. It is noticed that for a given M, (23)-(24) can always be satisfied as long as $|\lambda|$ is sufficiently large (or λ is sufficiently negative). Therefore we have

THEOREM. — *If λ is sufficiently negative, then*

$$\lambda\eta_1 + \alpha\eta_1^2 + \beta\eta_{1xx} = r(x), \qquad x_- < x < x_+ \tag{25}$$

$$\eta_1(x_-) = \eta_x(x_-) = 0 \tag{26}$$

has a classical solution.

Having established the existence of a solution of (20)-(21) from $-\infty$ to x_- and from x_- to x_+, we now extend the solution to $[x_+, \infty)$. Multiplying (20) by $\eta_{1x}(x)$ and integrating the resulting equation from x_+ to $x > x_+$, we have

$$\frac{3\beta}{2\alpha}(\eta_{1x})^2 = -\eta_1^3 - \frac{3\lambda}{2\alpha}\eta_1^2 + D \equiv P(\eta_1), \qquad x > x_+ \tag{27}$$

where

$$D = (\eta_1^+)^3 + \frac{3\lambda}{2\alpha}(\eta_1^+)^2 + \frac{3\beta}{2\alpha}(\eta_{1x}^+)^2, \tag{28}$$

$$\eta_1^+ = \eta_1(x_+),\ \eta_{1x}^+ = \eta_{1x}(x_+).$$

For a given geometry of the channel, D is a function of λ only. λ can be chosen to make $P(\eta_1)$ [*see* (27)] have three distinct zeros, a double zero, or only one real zero. Correspondingly, (27) has a cnoidal wave solution, a wave free solution (hydraulic fall), and an unbounded solution respectively ([Whitham, 1974] and [Benjamin Lighthill, 1954]). It is easy to show that if

$$D(\lambda) = \lambda^3/(2\alpha^3), \tag{29}$$

Then $P(\eta_1)$ has a double root. The solution of (29) is denoted as λ_L and is negative. Numerically, we found that $D(\lambda)$ is positive and bounded as $\lambda \to -\infty$. Hence (29) has a solution λ_L (see *Fig.* 3).

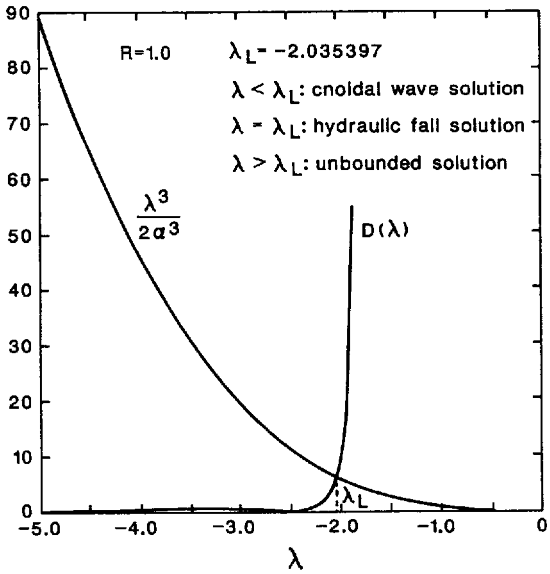

Fig. 3. – Graphic solution of (29) for the given obstruction.

$$r(x)=\begin{cases}\sqrt{1-x^2}, & |x|\leqq 1\\ 0, & |x|>1.\end{cases}$$

As $\lambda \leqq \lambda_L$, (27) has a solution

$$\eta_1(x)=\frac{\lambda}{\alpha}\left[\cos\left(\theta+\frac{4\pi}{3}\right)-\frac{1}{2}+\left(\cos\theta-\cos\left(\theta+\frac{4\pi}{3}\right)\right)\cdot cn^2\sqrt{\frac{\lambda}{6\beta}\left(\cos\theta-\cos\left(\theta+\frac{2\pi}{3}\right)\right)}(x-x_0)\right]. \tag{30}$$

The phase shift x_0 is in [0, T] and is determined by

$$\eta_1^+=\frac{\lambda}{2}\left[\cos\left(\theta+\frac{4\pi}{2}\right)-\frac{1}{2}+\left(\cos\theta-\cos\left(\theta+\frac{4\pi}{3}\right)\right)\cdot cn^2\sqrt{\frac{\lambda}{6\beta}\left(\cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)\right)}(x_+-x_0)\right]. \tag{31}$$

T is the period of the cnoidal wave of (30) and is given by

$$T=2K(k^2)/\sqrt{\frac{\lambda}{6\beta}\left(\cos\theta-\cos\left(\theta+\frac{2\pi}{3}\right)\right)}. \tag{32}$$

The parameter θ and k^2 are:

$$\theta=\frac{1}{3}\arccos(-1+4\alpha^3 D/\lambda^3), \qquad 0\leqq\theta\leqq\frac{\pi}{3}, \tag{33}$$

$$k^2 = \frac{\cos\theta - \cos(\theta + (4\pi/3)}{\cos\theta - \cos(\theta + (2\pi/3)} \leq 1. \tag{34}$$

K (k^2) is the complete elliptic integral defined by

$$K(k^2) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}. \tag{35}$$

The hydraulic fall solution is the limit of (30) as $\lambda\pi \uparrow \lambda_L$. In this case, (29) holds. By (33)-(34), $\theta = 0$, $k^2 = 1$. Since $K(1) = \infty$, the period $T = \infty$ by (32). (31) becomes

$$\eta_1(x) = (\lambda_L/\alpha)(-1 + (3/2)\operatorname{sech}^2 \sqrt{\lambda_L/(4\beta)}(x - x_0), \tag{36}$$

and x_0 is determined by

$$\eta_1^+ = (\lambda_L/\alpha)(-1 + (3/2)\operatorname{sech}^2 \sqrt{\lambda_L/(4\beta)}(x_+ - x_0). \tag{37}$$

As $x \to \infty$, $\eta_1(x) \to -\lambda_L/\alpha < 0$. Therefore there occurs a cascade of fluid in the channel flow. This is the new equilibrium we intended to find.

Remark. – *If* $\lambda \to -\infty$, *then* $\eta_1(x)$ *of* (31) *become sinusoidal waves as predicted by linear theory* [Lamb, 1945].

4. Numerical results

To find the profile of the free surface, we need to solve (20)-(21) numerically. This can be easily done by using an initial problem solver DVERK, which is a subroutine in IMSL (International Mathematical and Statistical Libraries). One can set initial point at $x = x_-$ and initial conditions as $\eta_1(x_-) = \eta_{1x}(x_-) = 0$.

To find the hydraulic fall solution, one first needs to find λ_L, which is determined by (29). To find λ_L, one can use a do-loop for λ and solve (25)-(26) for each λ still by DVERK. In this way a curve $D = D(\lambda)$ $(\lambda \leq 0)$ defined by (28) is obtained. Then the intersection of $D = D(\lambda)$ and $D = \lambda^3/(2\alpha^3)$ gives λ_L (see *Fig.* 3). Therefore, (i) when $\lambda < \lambda_L$, the downstream free surface consists of a cnoidal wave; (ii) when $\lambda = \lambda_L$, the downstream free surface is wave free; (iii) when $\lambda >_L$, the fluid flow can not reach an equilibrium.

The number of zeros of $P(\eta_1)$ determines the behavior of the downstream solution, while this number is determined by λ. Therefore the downstream solution is very sensible to the value of λ when λ is near λ_L. This is due to the situation that if $\lambda < \lambda_L$, the downstream solution is periodic and that if $\lambda > \lambda_L$, the downstream solution is unbounded. In our numerical examples, we computed λ_L upto 10^{-6} accuracy. This still does not allow us to compute the hydraulic solutions for a large x value (*i.e.* for downstream). We have to stop our computation at a certain place. Numerically, we observed that if we decrease λ_L by 10^{-6}, the downstream solution becomes periodic cnoidal wave and that if we increase λ_L by 10^{-6}, the downstream solution quickly blows up. The reason for this is actually quite simple, since a little computation error can change the number of zeros of $P(\eta_1)$, and hence change the behavior of the downstream solution. Nevertheless, the theory insures the existence of the hydraulic fall solution. So if we find a lower surface level at a reasonable distance downstream, we can stop our computation and take this as the hydraulic fall solution.

We have performed detailed computations for a triangular channel with $b=2$, $d=1$, and for a rectangular channel with $b=1$, $d=1$.

(i) Triangular channel:

$$r(x)=\begin{cases} R\sqrt{1-x^2}, & |x|\leqq 1 \\ 0, & |x|>1. \end{cases}$$

Three typical free surface profiles for $R=1.0$ and $\lambda=5.0$ (sinusoidal wave), $\lambda=2.05$ (cnoidal wave) and $\lambda=\lambda_L=-2.035397$ (hydraulic fall) are shown in Figure 4.

In order to see how the size of the obstruction affects the shape of the free surface, we plot the relationship of λ_L and R in Figure 5.

From Figure 5, $|\lambda_L|$ is proportional to the size of the obstruction. As the obstruction disappears, $\lambda_L=0$ and the only subcritical solution is zero. This is what we expected.

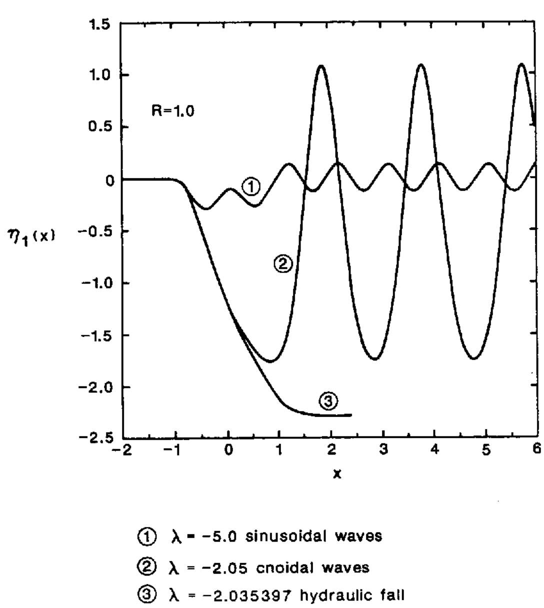

Fig. 4. – Three typical solutions of (20)-(21) for a triangular channel and

$$r(x)=\begin{cases} \sqrt{1-x^2}, & |x|\leqq 1 \\ 0, & |x|>1. \end{cases}$$

The period of the sinusoidal wave is approximately equal to $2\pi/\sqrt{-6\lambda}=2\pi/\sqrt{30}\approx 1.1471$.

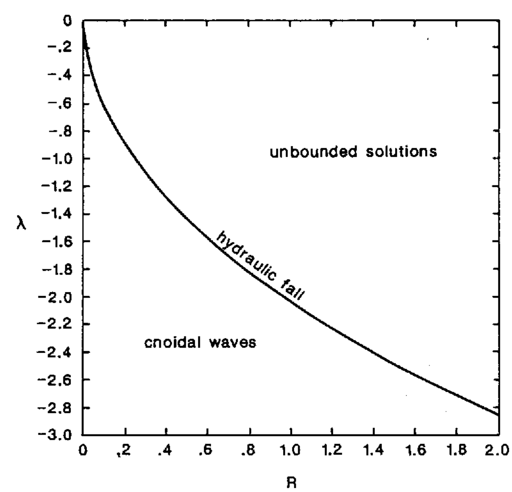

Fig. 5. – Relationship of λ_L and R for a triangular channel. R measures the size of the obstruction given by

$$r(x)=\begin{cases} R\sqrt{1-x^2}, & |x|\leqq 1 \\ 0, & |x|>1. \end{cases}$$

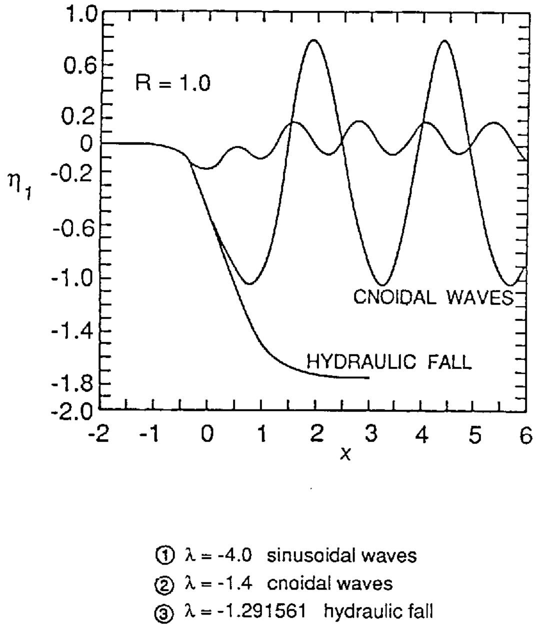

Fig. 6. – Three typical subcritical solutions of (20)-(21) for a rectangular channel and

$$r(x)=\begin{cases}\sqrt{1-x^2}, & |x|\leqq 1\\ 0, & |x|>1.\end{cases}$$

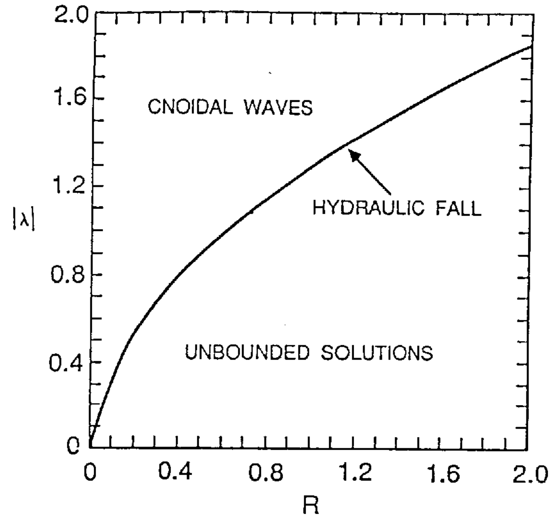

Fig. 7. – Relationship of λ_L and R for a rectangular channel. R measures the size of the obstruction given by

$$r(x)=\begin{cases}R\sqrt{1-x^2}, & |x|\leqq 1\\ 0, & |x|>1.\end{cases}$$

(ii) Rectangular channel:

$$r(x)=\begin{cases}R\sqrt{1-x^2}, & |x|\leqq 1\\ 0, & |x|>1.\end{cases}$$

Three typical free surface profiles for $R=1.0$ and $\lambda=4.0$ (sinusoidal wave), $\lambda=-1.4$ (cnoidal wave), and $\lambda=\lambda_L=-1.291561$ (hydraulic fall) are shown in Figure 6. The relationship between λ_L and R is shown in Figure 7.

APPENDIX

Derivation of a forced K-dV equation

In this appendix, we present the detail of the derivation of the forced $K-dV$ equation in a general situation. First of all, the time is included. Secondly, the upstream flow is not assumed to be uniform in a cross section of the channel. Let the x^*-axis be aligned along the longititudinal direction of the channel, the y^*-axis along the spanwise direction and z^*-axis vertically in the opposite direction to gravitation. The x^*, y^* plane is placed on the undisturbed free surface. The equation of the boundary of the channel is $h^*(x^*, y^*, z^*)=0$. The equation of free surface is denoted by $z^*=\eta^*(x^*, y^*, t^*)$ where t^* stands for the time coordinate and the superscript* denotes dimensional quantities. Then the equations of motion and boundary conditions are

(1′) $$u^*_{x^*}+v^*_{y^*}+w^*_{z^*}=0,$$

(2′) $$u^{*\prime}_{t^*}+u^*\,u^*_{x^*}+v^*\,u^*_{y^*}+w^*\,u^*_{z^*}=-\frac{1}{\rho^*}p^*_{x^*},$$

(3′) $$v^*_{t^*}+u^*\,v^*_{x^*}+v^*\,{}^*_{y^*}+w^*\,v^*_{z^*}=-\frac{1}{\rho^*}p^*_{y^*},$$

(4′) $$w^*_{t^*}+u^*\,w^*_{x^*}+v^*\,w^*_{y^*}+w^*\,w^*_{z^*}=-g-\frac{1}{\rho^*}p^*_{z^*};$$

on the free surface

$$z=\eta^*(x^*, y^*, t^*),$$

(5′) $$\eta^*_{t^*}+u\,\eta^*_{x^*}+v^*\,\eta^*_{y^*}-w^*=0,$$

(6) $$p^*=\bar{p}^*(x^*);$$

on the wall of the channel

$$h^*(x^*, y^*, z^*)=0,$$

(7′) $$u^*\,h^*_{x^*}+v^*\,h^*_{y^*}+w^*\,h^*_{z^*}=0.$$

Here (u^*, v^*, w^*) is velocity; ρ^* is density; p^* is pressure; g is the gravitational acceleration constant; and $\bar{p}^*$, which is assumed to be function of only x^*, is the disturnance pressure on the free surface (see *Fig.* 1).

To nondimensionalize (1′)-(7′), the following dimensionless quantities are introduced.

$$\varepsilon=\left(\frac{\mathrm{H}}{\mathrm{L}}\right)^2 \ll 1, t=\varepsilon^{3/2}\sqrt{\frac{g}{\mathrm{H}}}\,t^*,$$

$$(x, y, z)=\frac{1}{\mathrm{H}}(\varepsilon^{1/2}x^*, y^*, z^*),$$

$$\eta=\frac{\eta^*}{\mathrm{H}}, \qquad p=\frac{p^*}{\rho^* g\mathrm{H}}, \qquad \bar{p}=\varepsilon^2\frac{\bar{p}^*}{\rho g \mathrm{H}},$$

$$(u, v, w)=\frac{1}{\sqrt{g\mathrm{H}}}(u^*, \varepsilon^{-1/2}v^*, \varepsilon^{-1/2}w^*),$$

$$h_1=\varepsilon^{-5/2}h^*_{x^*}, h_2=h^*_{y^*}, h_3=h^*_{z^*}.$$

In terms of these dimensionless quantities and by approximating the boundary conditions on the free surface around $z=0$, (1′)-(7′) can be expressed as

(8′) $$u_x+v_y+w_z=0,$$

(9′) $$\varepsilon u_t + u u_x + p_x + v u_y + w u_z = 0,$$

(10′) $$\varepsilon v_t + \varepsilon u v_x + p_y + v v_y + w v_z = 0,$$

(11′) $$\varepsilon w_t + \varepsilon u w_x + 1 + p_z + v w_y + w w_z = 0;$$

on $z=0$,

(12′) $$w - \varepsilon \eta_t - u \eta_x - v \eta_y = 0,$$

(13′) $$p = \varepsilon^2 \bar{p} + \eta;$$

on $h=0$,

(14′) $$\varepsilon^2 u k_1 + v h_2 + w h_3 = 0.$$

For an upstream current $U_0(y, z) = u_0(y, z) + \varepsilon\lambda + O(\varepsilon^2)$, we assume an asymptotic expansion of the following form:

(15′) $$(u, v, w, \eta, p) = (u_0(y, z), 0, 0, 0, -z) + \varepsilon(u_1 + \lambda, v_1, w_1, \eta_1, p_1) + \varepsilon^2(u_2, v_2, w_2, \eta_2, p_2) + O(\varepsilon^3).$$

Inserting (15′) into (8′)-(14′) and assembling the resulting equations according to the powers of ε, it follows that the equations of the order ε order ε and ε^2 are as below.
$O(\varepsilon)$:

(16′) $$u_{1x} + v_{1y} + w_{1z} = 0,$$

(17′) $$u_0 u_{1x} + p_{1x} + v_1 u_{0y} + w_1 u_{0x} = 0,$$

(18′) $$p_{1y} = 0,$$

(19′) $$p_{1z} = 0;$$

on $z=0$,

(20′) $$w_1 - u_0 \eta_{1x} = 0, \qquad p_1 = \eta_1;$$

on $h=0$

(21′) $$v_1 h_2 + w_1 h_3 = 0.$$

$O(\varepsilon^2)$:

(22′) $$u_{2x} + v_{2y} + w_{2z} = 0,$$

(23′) $$u_{1t} + u_0 u_{2x} + (u_1 + \lambda) u_{1x} + p_{2x} + v_1 u_{1y} + v_2 u_{0y} + w_1 u_{1z} + w_2 u_{0z} = 0,$$

(24′) $$u_0 v_{1x} + p_{2y} = 0,$$

(25′) $$u_0 w_{1x} + p_{2z} = 0,$$

on $z=0$,

(26′) $$w_2 - \eta_{1t} - u_0 \eta_{2x} - (u_1 + \lambda) \eta_{1x} - v_1 \eta_{1y} = 0,$$

(27′) $$p_2=\bar{p}+\eta_2;$$

on $h=0$,

(28′) $$v_2 h_2+w_2 h_3=-u_0 h_1.$$

From (16′)-(21′), one can derive that (see *Fig.* 2)

$$\left(\int_{D}\int\frac{dy\,dz}{u_0^2(y,z)}-\int_{\Gamma}ds\right)\eta_{1x}=0.$$

For nontrivial solutions, $\eta_{1x}\neq 0$. It follows the dispersion relation,

(29′) $$\int_{D}\int\frac{dy\,dz}{u_0^2(y,z)}=b.$$

Any upstream current $U_0(y, z)$ with $u_0(y, z)$ satisfying (29′) is called a critical flow. By (24′)-(25′),

$$\left(\frac{p_{2y}}{u_0^2}\right)_y+\left(\frac{p_{2z}}{u_0^2}\right)_z=-\frac{p_{1xx}}{u_0^2}.$$

Assume

(30′) $$p_2=-\varphi(y,z)\,p_{1xx}(x,t)+C_1(x,t),$$

then φ satisfies

(31′) $$\nabla\left(\frac{\nabla\varphi}{u_0^2}\right)=\frac{1}{u_0^2}\quad\text{in D},$$

(32′) $$\varphi_z=u_0^2\quad\text{on }\Gamma,$$

(33′) $$\varphi_n=0\quad\text{on C}$$

where $\nabla=(\partial/\partial y,\ \partial/\partial z)$ and φ_n is the outward normal derivative of φ on C.

By (16′),

(34′) $$(u_1,v_1,w_1)=\left(-\nabla\left(\frac{\nabla\varphi}{u_0}\right)p_1,\ \frac{\nabla\varphi}{u_0}p_{1x}\right).$$

Multiplying (22′) by u_0^{-1} and (23′) by $-u_0^{-2}$, and integrating the sum of the resulting equations over D, it follows from (29′), (30′) and (34′) that

(35′) $$m_1\eta_{1t}+\lambda m_1\eta_{1x}+m_2\eta_1\eta_{1x}+m_3\eta_{1xxx}=-f(x)$$

where

$$m_1 = -2 \iint_D \frac{dy\,dz}{u_0^3}, \tag{36'}$$

$$m_2 = 3 \iint_D \frac{dy\,dz}{u_0^4} - \int_\Gamma \left(\frac{\varphi_y}{u_0^2}\right)_y ds, \tag{37'}$$

$$m_3 = \iint_D \left|\frac{\nabla\varphi}{u_0}\right|^2 dy\,dz, \tag{38'}$$

$$f(x) = b\bar{p}_x + h_1(x) \int_C \frac{ds}{\sqrt{h_2^2 + h_3^2}}. \tag{39'}$$

Equation (35') is the forced K-*d*V equation we desired to derive. To determine m_1, m_2 and m_1, one needs to solve the Neumann problem (31')-(33').

For the case of steady state and uniform upstream current, $\eta_{1t}=0$ and $u_0(y, z)=u_c=$constant [*see Eq.* (10)]. Then (35')-(39') reduce to (11)-(15).

In the above derivation, we have assumed that the cross section of the channel varies at order $O(\varepsilon^2)$ in the longitudinal direction of length $O(1)$ and the upstream ambient shear flow is near critical. The resulting equation is the forced K-*d*V equation (35'). One may be curious to know how the flow phenomena change for different obstruction sizes and vanishing surface pressure. There are research works which studied $O(\varepsilon^3)$ and $O(\varepsilon)$ order variation of cross sections of the channels. When the cross section of a channel varies at order $O(\varepsilon^3)$ in the longitudinal direction over a length $O(1)$ and the upstream ambient shear flow is near critical, then the governing equation of the first order elevation of the free surface, derived first by Peters [1966], is a K-*d*V equation. This equation possesses traveling soliton solutions. When the cross section of a channel varies at order $O(\varepsilon)$ in the longitudinal direction over a length $O(1)$, Shen [1975] derived a governing equation for the first order elevation of the free surface, which is an equation of K-*d*V type with variable coefficients. This equation has been successfully used to study soliton fissions in channels [Zhong & Shen, 1982].

REFERENCES

AKYLAS T. R. 1984, On the Excitation of Long Nonlinear Water Waves by a Moving Pressure Distribution, *J. Fluid Mech.*, **141**, 455-466.

BAINES P. G. 1987, Upstream Blocking and Airflow over Mountains, *Ann. Rev. Fluid Mech.*, **19**, 75-97.

BENJAMIN T. B., LIGHTHILL M. J. 1954, On Cnoidal Waves and Bores, *Proc. R. Soc. Lond. A*, **224**, 448-460.

COLE S. L. 1985, Transient Waves Produced by Flow Past a Bump, *Wave Motion*, **7**, 579-587.

ERTEKIN R. C., WEBSTER W. C., WEHAUSEN J. V. 1986, Waves caused by a Moving Disturbance in a Shallow Channel of Finite Width, *J. Fluid Mech.*, **169**, 275-292.

FORBES L. K. 1988, Critical Free-Surface Flow over a Semi-circular Obstruction, *J. Engng. Math.*, **22**, 3-13.

FORBES L. K., SCHWARTZ L. W. 1982, Free-surface Flow over a Semi-circular Obstruction, *J. Fluid Mech.*, **114**, 299-314.

GRIMSHAW R. 1987, Resonant Forcing of Barotropic Coastally Trapped Waves, *J. Phys. Oceanogr.*, **17**, 53-65.

GRIMSHAW R., SMYTH N. 1986, Resonant Flow of a Stratified Fluid over Topography, *J. Fluid Mech.*, **169**, 429-464.

HUANG DE-BO, SIBUL O. J., WEBSTER W. C., WEHAUSEN J. V., WU DE-MING, WU T. Y. 1982, Ships Moving in the Transcritical Range, Conf. on Behavior of Ships in Restricted Waters, *Varna: Proc.*, **2**, 26/1-26/10.

LAMB H. 1945, *Hydrodynamics*, 6th ed., Dover Publications, New York.

MEI C. C. 1986, Radiation of Solitons by Slender Bodies Advancing in a Shallow Channel, *J. Fluid Mech.*, **162**, 53-67.

PETERS A. S. 1966, Rotational and Irrotational Solitary Waves in a Channel with Arbitrary Cross Section, *Comm. Pure Appl. Math.*, **19**, 445-471.

SHEN M. C. 1975, Ray Method for Surface Waves on Fluid of Variable Depth, *SIAM Rev.*, **17**, 38-55.

SHEN S. S. P., SHEN M. C., SUN S. M. 1989, A Model Equation for Steady Surface Waves over a Bump, *J. Engng. Math.*, in press.

STOCKER J. J. 1957, *Water Waves*, Interscience Publishers, New York.

VANDEN-BROECK J. M. 1987, Free-surface Flows over an Obstruction in a Channel, *Phys. Fluids*, **30**, 2315-2317.

WHITHAM G. B. 1974, Linear and Nonlinear Waves, John Wiley & Sons, New York, WU T. YAO-TSU 1987, Generation of Upstream Advancing Solitons by Moving Disturbances, *J. Fluid Mech.*, **184**, 75-99.

WU DE-MING, WU T. Y. 1982, Three-dimensional Nonlinear Long Waves due to Moving Surface Pressure, Proc. 14th Symp. Naval Hydrodynam., *Ann. Arbor, Mich.*, 103-129.

YIH CHIA-SHUN 1979, Fluids Mechanics, West River Press, *Ann. Arbor*.

ZHONG, X. C., SHEN M. C. 1982, Water Waves in a Channel of Variable Cross Section, *MRC Tech. Rep.* #2355, University of Wisconsin-Madison.

(Manuscript received September 28, 1989.)