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COLLISION OF UNIFORM SOLITON TRAINS IN ASYMMETRIC SYSTEMS

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Abstract. This paper describes the original discovery of soliton collisions governed by the forced Korteweg-de Vries equation and forced nonlinear Schrödinger equation, respectively. These two nonlinear dynamic systems do not have infinitely many conservation laws and are grouply asymmetric due to external forcing. The forcing makes it possible to generate a train of solitary waves of the same size. The traditional group-theoretical method is no longer appropriate for describing these solitary waves. This paper numerically demonstrates that the collision process of the solitary waves generated by the forcing in the two asymmetric dynamic systems, hence confirms that the solitary waves are solitons.

Keywords. Forced solitons, soliton collision, forced Korteweg-de Vries equation, forced nonlinear Schrödinger equation, numerical method.
AMS (MOS) subject classification: 35Q51, 76C10

1 Introduction

In the milestone paper on modern nonlinear science, Zabusky and Kruskal (1965) studied Fermi, Pasta and Ulam's discovery of recurrence of energy distribution and showed that certain traveling waves can pass each other and retain their original shapes [8]. These waves are called "solitons" (i.e., lonely particles) and they have produced a historically stronger than ever stimulation of the research into the nonlinear-wave mathematics with symmetries. These symmetries are equivalent to the existence of infinitely many conservation laws. Various beautiful mathematics has been generated from these symmetries, such as soliton-hierarchy in Lie algebra, shape transition in geometry, Backlund transform, and inverse-scattering method. Unfortunately, when certain symmetries, such as the translation invariant property, are broken, or certain conservation laws, such as the conservation of momentum, are not satisfied, the above mathematics is no longer working. In the practical world, asymmetry is common and absolute symmetry is rare. An outstanding question is that: are there still solitons, under the collision definition of Zabusky and Kruskal, governed by asymmetric dynamic systems?

The first evidence of the existence of such solitons was provided by the celebrated discovery of upstream radiated waves by a Caltech fluid mechanics group led by T. Y. Wu in 1982 [cf. [1] and [7] and references therein]. Wu considered the following nonlinear water waves. The rest water in a two-dimensional, long channel obtains mechanical energy from a moving bump on the bottom of the channel. When the bump moves to the left at a speed near the critical shallow water wave velocity, solitary surface waves are periodically generated at the bow side of the bump and radiated away from the bump to the upstream. They claimed that these solitary waves are solitons. These waves have been successfully modeled by the forced Korteweg-de Vries equation (fKdV) (Shen, 1993, 1996). The current paper demonstrates that these waves can go through a collision test. It hence validates Wu's claim of solitons under the definition of Zabusky and Kruskal.

The second model is the forced nonlinear Schrödinger equation (fNLS). In 1994 at the International Conference of Differential Equations and Control Theory, the result of soliton radiation in two directions based upon the fNLS was announced (see Fig. 3 of [3]). The amplitude of the solution of the fNLS oscillates at the forcing site and radiates a single soliton to the positive x-direction and an identical one to the negative x-direction. Again it is still to be demonstrated that the radiated solitons can pass the collision test, which is another purpose of this purpose.

This paper is arranged as follows. Section 2 describes the approximate solitary wave solutions of the fKDV. Section 3 presents the numerical simulations of soliton collision process in both fKdV and fNLS. Conclusion and discussion are given in Section 4.

2 Approximate solutions of fKdV

To gain some idea about the solution behavior of the fKdV, we first investigate approximate solutions to the fKdV model:

$$(1) \qquad \eta_t + \lambda\eta_x - \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxx} = \frac{P}{2}\delta_x(x), \qquad -\infty < x < \infty.$$

Here $\eta(x,t)$ describes the free surface profile in the Wu model of water flows over a bump, λ measures the deviation of the bump speed from the shallow water velocity, P is computed from the cross section area of the bump, $\delta(x)$ is the Dirac delta function, x is the spatial coordinate along the channel, and t is time. The control parameters in this model are the bump size parameter P and the bump

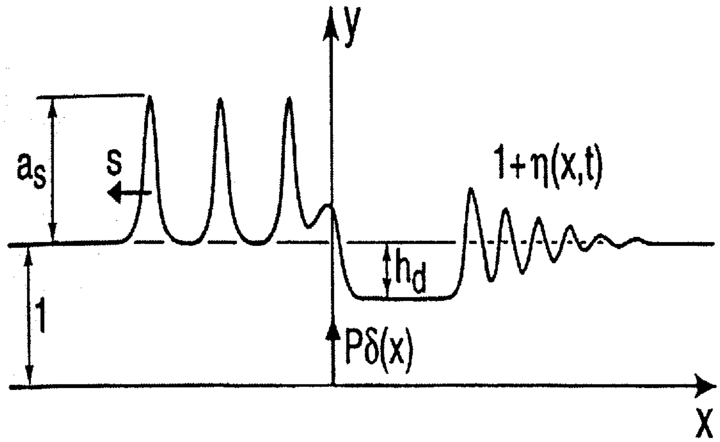

Figure 1: An illustration of the schematic solution $\eta(x,t)$ of fKdV for a fixed time t.

speed parameter λ. The initial condition for Equation (1) is always $\eta(x,0) = 0$, i.e. the rest water. The solution consists of a soliton region upstream, a depression region immediately on the lee side of the bump and a lee wave further downstream. The schematic solution is shown in Figure 1. The kth upstream soliton is governed by

(2) $$\eta^{(k)}(x,t) = a_s \operatorname{sech}^2 \left\{ (\sqrt{3a_s}/2)(x + st - x_k) \right\},$$

where x_k is the specific phase shift for the kth soliton. Based upon the mass balance postulate that the upstream soliton mass comes solely from the downstream depression when time is sufficiently large, one can derive approximate expressions of the depression depth h_d, soliton amplitude a_s, soliton propagation speed s, and soliton generation period T_s in terms of the control parameters P and λ (cf. [3] and [5]):

(3) $$h_d = \left(\frac{3}{4}P^2\right)^{1/3} - \frac{2}{3}\lambda,$$

(4) $$a_s = \frac{2(h_d + \frac{4}{3}\lambda)(h_d + \frac{1}{3}\lambda)}{h_d},$$

(5) $$s = \frac{a_s}{2} - \lambda,$$

(6) $$T_s = \frac{16}{3}\left[\frac{2(h_d + \frac{1}{3}\lambda)}{3h_d^3(h_d + \frac{4}{3}\lambda)}\right]^{1/2}.$$

Since these are asymptotic approximations as time t approaches infinity, they deviate from numerical solutions by a small error for the cases in the present work. When P and λ are constants, the upstream solitons are identical, form a uniform soliton train and move at the same speed, thus, one cannot see the collision process. To make soliton collision happen, an idea is to generate and isolate two trains of uniform solitons of different amplitudes. The results from [4] and [6] show that soliton isolation can be made. When studying the instability of multiple fKdV solitary waves, it is shown in Figure 3 of [6] that the higher unstable solitary wave, while shrinking to the lower stable stationary solitary wave, radiates away a single soliton upstream and generates a lee wave far downstream. From this observation and the conservation of mass, one may think that the number of solitons generated by the forcing depends on the initial condition, bump speed and bump size. Ref. [2] considered the case of non-constant bump speed and observed the resonance of the upstream solitons. In the present work, the case of the variable bump size is investigated and the soliton collision process is clearly demonstrated: the train of uniform solitons of the larger amplitude moves faster and overpasses the smaller one, and all the solitons retain their original shapes after the collision.

3 Numerical simulations for soliton collision

The initial value problems of the fKdV and fNLS are solved by a semi-implicit pseudo-spectral method. The Fourier transform is made for x from $-L$ to L, where L is taken sufficiently large to avoid boundary reflection. The integration over t is by the leap-frog method. Special attention is paid to the nonlinear and dispersion terms. The details of the method can be found in Chapter 6 of Ref. [3].

The following parameters are adopted for the fKdV soliton collision process

$$\lambda = 0, \tag{7}$$

$$P = \begin{cases} 0.4, & 0 \le t < 26.5 \\ 0.0, & 26.5 \le t < 38 \\ 1.0, & 38 \le t < 49 \\ 0.0, & 49 \le t, \end{cases} \tag{8}$$

$$\eta(x,0) = 0, \qquad \eta(\pm\infty, t) = \eta_x(\pm\infty, t) = 0. \tag{9}$$

The numerical solution of the differential equation (1) with the above data is shown in Figure 2. The depth of the

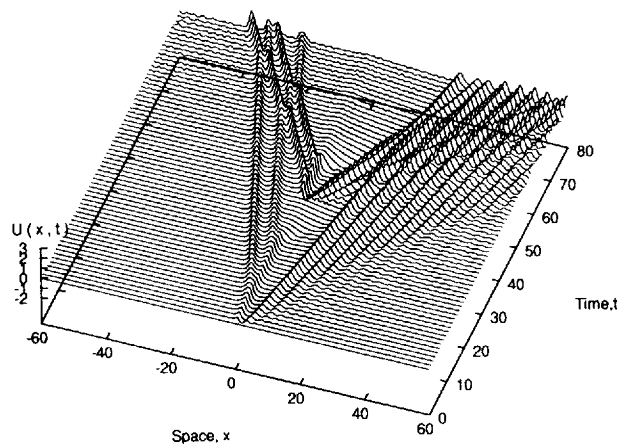

Figure 2: Numerical solution of the fKdV (1) with conditions (7), (8) and (9).

depression h_d, and the amplitude a_s, speed s and generation period T_s of the mature solitons can be analytically estimated from (3), (4), (5) and (6) for $\lambda = 0$ and $P = 0.4$:

$$h_d = 0.4932, \quad a_s = 0.9865, \quad s = 0.4932, \quad \text{and} \quad T_s = 12.57.$$

Since $t = 26.5 > 2T_s = 25.04$, two solitons are generated. The numerical solution shows that the first one has an amplitude 0.9767, which is very close to the above analytic approximation 0.9865, and is considered completely mature. The second one is almost mature at $t = 26.5$. A third soliton would be generated if we wait for the second one to become completely mature. Since we intend to include only two solitons in the collision test, the forcing $P = 0.4$ is thus cut off at $t = 26.5$. Because there is no further mass supply for $26.5 < t < 38$, no more solitons are generated during this time period. The near rectangular depression region on the immediate lee side of the bump, when the forcing was in action, gradually becomes triangular (See Figures 1 and 3).

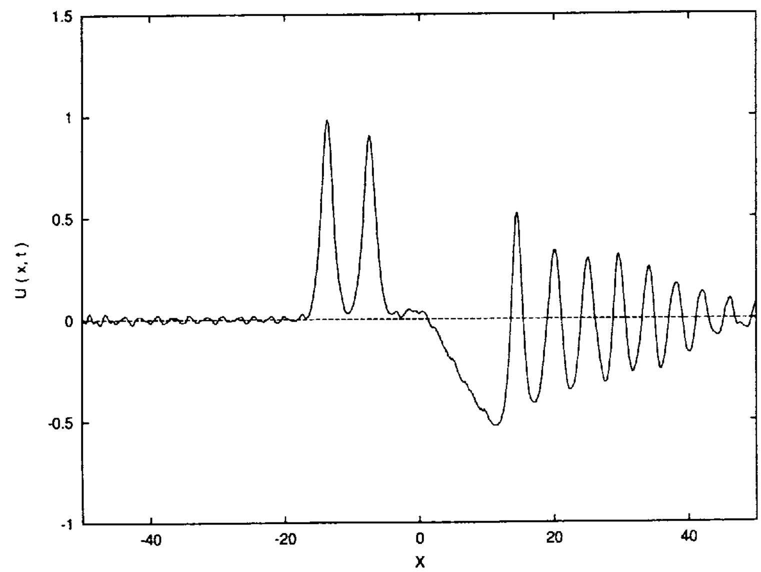

Figure 3: The same solution as shown in Fig. 2 but at $t = 38$.

The forcing is increased to $P = 1.0$ at $t = 38$ and remains at this strength during $38 < t < 49$. The results from the formulas (3) -(6) for $\lambda = 0$ and $P = 1.0$ are

$$h_d = 0.9086, \quad a_s = 1.8172, \quad s = 0.9086, \quad \text{and} \quad T_s = 5.0280.$$

Two more solitons are generated during this period because its length of 11 is a bit longer than twice of $T_s = 5.0280$. The numerical solution shows that the first soliton is completely mature with an amplitude of 1.8024, and the second one is almost mature. The forcing is reduced to zero after $t = 49$. The two larger solitons move faster, collide with and then overpass the smaller ones generated earlier. All the solitons retain their original shapes after the collision. The process clearly demonstrates that the solitary waves generated by the fKdV are indeed the solitons under the definition of Zabusky and Kruskal. The second part of this section is to communicate the soliton collision results for the fNLS equation:

$$iu_t + \gamma u_{xx} + \mu |u|^2 u = P\delta(x), \tag{10}$$

where γ, μ and P are control parameters, $i = \sqrt{-1}$ is the imaginary unit, and the function $u(x,t)$ is, of course, complex valued. In Ref. [4], $\gamma = 1$, $\mu = 2$ and $P = 1.2$ are fixed. It was observed that two identical solitary waves of $|u(x,t)|$ are moving in opposite directions. At the forcing site $x = 0$, the function $|u(0,t)|$ is periodically oscillatory and this oscillation is maintained by the forcing. The amplitude and frequency of the $|u(0,t)|$ oscillation are increasing functions of P, and so are the amplitude and the speed of the radiated solitons. When the forcing is set to zero at a certain time, the $|u(0,t)|$ oscillation stops and the solitons keep moving at their original speed. But the amplitudes of the solitons appear to oscillate slowly with respect to time. This oscillation is much slower than that of $|u(0,t)|$. To show that the radiated fNLS solitary waves are solitons, similar to the fKdV case, we generate and isolate a larger soliton from a smaller one and let them go through a collision process. To do so, P now varies as a function of t. The following conditions are specified:

$$\gamma = 1, \qquad \mu = 2, \tag{11}$$

$$P = \begin{cases} 0.44, & 0 \le t < 28.5 \\ 1.2, & 28.5 \le t < 60 \\ 0, & t \ge 60, \end{cases} \tag{12}$$

$$u(x,0) = 0, \qquad u(\pm\infty, t) = 0. \tag{13}$$

The amplitude $|u(x,t)|$ of the numerical solution of the fNLS (10) with the above conditions is shown in Figure 4.

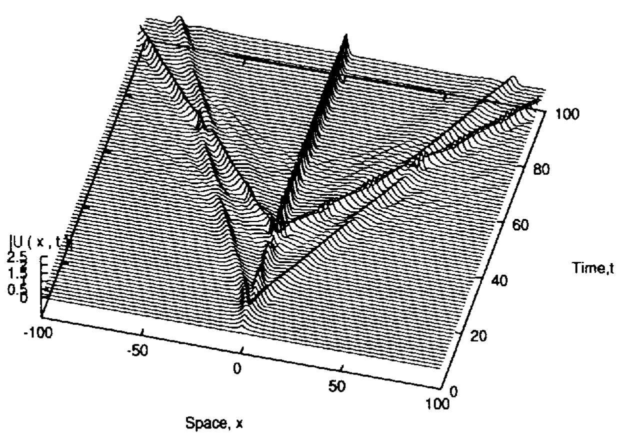

Figure 4: Numerical solution of $u(x,t)$ for the fNLS (10) with conditions (11), (12) and (13).

One can see that the solitary waves involved in the collision retain their original shapes after collision. Thus we conclude that the radiated solitary waves discovered in [4] are solitons.

4 Conclusion and discussion

Numerical simulations demonstrate that the fKdV can generate solitary waves of the same size. Whether the solitary waves are solitons has been successfully examined by a collision process. However, the fNLS generates only two solitons propagating in different directions. It remains to be investigated why the equation cannot generate more than two solitons. It is well known that the sine-Gordon equation can also generate solitons. We have attempted to obtain soliton solutions for the forced sine-Gordon equation, but so far we have not been successful. Therefore, whether fNLS can generate more than two solitons and whether the forced sine-Gordon equation has a soliton solution are still to be investigated.

5 Acknowledgments

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6 References

[1] S. J. Lee, G. T. Yates, and T. Y. Wu, Experiments and analyses of upstream-advancing solitary waves generated by moving disturbances, *J. Fluid Mech.*, **199**, (1989) 569-593.

[2] L.G. Redekopp and Z. You, Passage through resonance for the forced Korteweg-de Vries equation, *Phys. Rev. Lett.*, **74**, (1995) 5158-5161.

[3] S.S. Shen, A Course on Nonlinear Waves, Kluwer Academic Publisher, Boston, 1993.

[4] S.S. Shen, Forced evolution equations and detection of climate change, in *Differential Equations and Control Theory* (Proceedings of the International Conference on Differential Equations and Control Theory, May 1994, Wuhan), edited by Z. Deng, Z. Liang, G. Lu and S. Ruan, Marcel Dekker, New York, 1996.

[5] S. S. Shen, Energy distribution for waves in transcritical flows over a bump, *Wave Motion*, **23**, (1996) 39-48.

[6] S.S. Shen, R.P. Manohar and L. Gong, Stability of the lower cusped solitary waves, *Phys. Fluids*, **A 7**, 2507-2509 (1995). Phys. Fluids A 7, 2507 (1995).

[7] T. Y. Wu, Generation of upstream advancing solitons b moving disturbances, *J. Fluid Mech.*, **184**, (1987) 75-100.

[8] N.J. Zabusky and M.D. Kruskal, Interaction of "solitons" in a collisionless plasma and the recurrence of the initial states, *Phys. Rev. Lett.*, **15**, (1965) 240-243.

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