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INTERPRETATION OF THE STABILITY AND INSTABILITY OF THE SOLITARY WAVES GOVERNED BY A FORCED KORTEWEG-DE VRIES EQUATION

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ABSTRACT. In view of the maximum height of a solitary shallow-water wave in a channel, this paper provides an interpretation of the stability and instability of the solitary waves governed by a forced Korteweg-de Vries equation. The interpretation implies Malomed's conjecture [**3**]: of the two cusped solitary waves of a locally forced Korteweg-de Vries equation, the lower one is stable. Numerical simulations show that the higher solitary wave degenerates into the lower, stable solitary wave and radiates a soliton upstream and a wake down stream. This is a KdV soliton but is not a stable water-surface profile because its amplitude is higher than the unstable solitary wave.

1. Introduction. In 1987, Vanden-Broeck [**1**] found that the stationary Euler equations for the water motion in a channel with a bump have two solitary-wave solutions, one higher than the other. Shen [**2**] showed that these two solitary waves can be approximately modeled by a forced Korteweg-de Vries equation (fKdV). Using the method of dynamical systems, Malomed [**3**] proved that the higher solitary wave is unstable. He also conjectured that the lower solitary wave is stable and claimed that the proof of this stability appears difficult. The cusped solitary wave solution of an fKdV found by Miles [**4**] corresponds to the Vanden-Broeck's lower solitary wave. The instability proof and the stability conjecture were substantiated by the numerical simulations of Shen et al. [**5**]. Their numerical simulations clearly demonstrated that the lower solitary wave is stable and the higher one is unstable. The mathematical proofs are complicated. The remaining question is as follows: can one intuitively explain the stability or instability of the solitary waves, without providing mathematical proof, numerical simulation and laboratory experiment? The purpose of this paper is to provide an interpretation of the stability and instability of solitary waves in view of the maximum height of a solitary wave. In addition,

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we conclude that the higher solitary wave radiates only one soliton upstream when it degenerates into the lower solitary wave. This soliton is a solution of free KdV equation. Although free KdV solitons are stable in the sense of the perturbation for KdV equation, this soliton is not a stable water-surface profile upstream because its amplitude is higher than the unstable solitary wave.

A brief summary of the history of solitary-wave research will be helpful. In 1834, John Scott Russell observed a stable solitary wave supported by shallow water in a channel. It was a "rounded, smooth and well defined heap of water." The wave was about 1-1/2 feet high and 30 feet long, and moving at about eight miles per hour. Russell chased the wave for one or two miles, and the wave finally diminished, perhaps due to dissipation. Suppose that Russell chased the wave for only a mile. As a result, the solitary wave lasted for seven or eight minutes. According to modern nonlinear wave theory, the depth of the channel, not reported in Russell's original description, should have been about five feet. Hence, the amplitude of the solitary wave observed by Russell was about 30 percent of the depth. It is not a difficult task to generate a single solitary wave of $0.3H$ in a channel in a modern fluid mechanics laboratory, where H is the depth of water. In 1974, Hamack and Segur [**6**] reported an observation of a solitary wave's height of $0.347H$. However, it is difficult to generate a very high solitary wave, say $0.7H$. Further, the height of a solitary wave has an upper limit. For a flat channel, Hunter and Vanden-Broeck [**7**] showed that the maximum height of the solitary wave can only be $0.83322H$, which corresponds to the upstream Froude number 1.29091.

Intuitively, one can accept that a free surface of water in a channel is either a flat surface or a periodic wave when the surface is smooth. However, it has been mathematically proven that the Euler equations of water motion have stable and smooth solitary-wave solutions when the upstream Froude number is in a particular range. This proof implies that the flat-surface solution of the Euler equations under the same ambient conditions is unstable. This implication is counter-intuitive, but has been widely accepted in the nonlinear-wave community.

By numerically solving Euler equations, Vanden-Broeck [**1**] found the two non-zero solutions when a bump is present at the bottom of a channel and the upstream Froude number is sufficiently large. Both free surfaces of the two solutions are smooth solitary waves. Because of the presence of the bump and non-zero Froude number, the flat surface is no longer a free-surface allowed by the Euler equations. Vanden-Broeck pointed out that the higher solitary wave can be regarded as a perturbation of the flat-channel solitary wave and the lower solitary wave as a perturbation of the zero solution of the flat bottom model. The perturbation is, of course, due to the presence of the bump. Since the solitary wave in a flat channel is stable, one might deduce that the higher solitary wave is stable. However, this deduction is incorrect.

This paper is arranged as follows. Section 2 recapitulates the fKdV equation and its correspondence to the physical quantities in a laboratory. The bifurcations of the stationary fKdV are shown in Section 3. Section 4 discusses the existing results for the maximum height of solitary waves and explains the instability of the higher solitary wave. Conclusions are included in Section 5.

2. Forced KdV equation and free surface waves over a bump. In a two-dimensional channel flow, a bump is at the bottom of the channel. The far upstream velocity is constant. The free-surface wave's profile is determined by the upstream velocity, the size and, possibly, the shape of the bump [**8**]. For a given bump, the waves are solely determined by the upstream velocity. When the velocity is very large, the water flow becomes a jet, and its impact with the bump results in a loss of much momentum and leads to a downstream turbulence. When the velocity is very slow with zero as its extreme case, the free surface will be flat. The former case is too complicated to be modeled by Euler equations, and the latter is trivial. A nontrivial, intriguing and interesting situation occurs when the velocity is near the shallow water speed. Physically, this is not surprising since the shallow water speed is the characteristic speed of small amplitude waves propagating on a water surface. Hence, the shallow water speed is analogous to the sound speed in the air. Therefore, the speed of the carrier wave near the shallow water speed is natural and common.

The physical setup of the problem and its mathematical model are as follows. The reference frame (x^*, y^*) is fixed on the bump whose profile is described by $h^*(x^*) = \varepsilon^2 H h(x)$, where the parameter ε is $\varepsilon = (\|h\|_\infty/H)^{1/2}$. Here $\|h\|_\infty$ is the maximum height of the bump. The free surface is assumed to be $\eta^* = \varepsilon H \eta_1(x,t)$, where $x = ((\|h\|_\infty/H)^{1/4})x^*/H$ is the dimensionless horizontal length scale, and $\eta_1(x,t)$ is governed by the forced Korteweg-de Vries equation (fKdV) below. The upstream velocity is $c^* = (gH)^{1/2}(1+\varepsilon\lambda)$, which is a perturbation from the shallow water speed $(gH)^{1/2}$. The function $\eta_1(x,t)$ satisfies an fKdV,

$$\eta_{1t} + \lambda\eta_1 x - \frac{3}{2}\eta_1\eta_{1x} - \frac{1}{6}\eta_{1xxx} = \frac{h_x(x)}{2}, \quad -\infty < x < \infty. \tag{1}$$

This is a mathematical model for the surface wave over a bump. When the bump is short, the forcing function $h(x)$ can be approximated by $P\delta(x)$, and $P = \varepsilon^{-3/2}S/H^2$ is the dimensionless area of the bump, S is the area of the bump, and $\delta(x)$ is the Dirac delta function.

3. Bifurcation and multiple solutions of the stationary fKdV. The stationary fKdV is

$$\lambda\eta_{0x} - \frac{3}{2}\eta_0\eta_{0x} - \frac{1}{6}\eta_{0xxx} = \frac{P}{2}\delta_x(x), \quad -\infty < x < \infty, \tag{2}$$

$$\eta_0(\pm\infty) = \eta_{0x}(\pm\infty) = \eta_{0xx}(\pm\infty) = 0 \tag{3}$$

where

$$P = \int_{-\infty}^{\infty} h(x)\,dx \tag{4}$$

is the area of the bump in the nondimensional space. The relationship between P and the area of the bump in the laboratory coordinates S is

$$P = \frac{S}{(\|h\|_\infty^3 H^5)^{1/4}}. \tag{5}$$

This boundary value problem does not have a solution when $\lambda = 0$. As a matter of fact, it also does not have a solution when $\lambda < \lambda_C$ where

$$\lambda_C = \frac{3}{4}\left(\frac{3}{2}\right)^{1/3} P^{2/3}. \tag{6}$$

When $\lambda > \lambda_C$, it has two cusped solitary wave solutions given below:

$$\eta_0(x) = 2\lambda\text{sech}^2\left[\sqrt{\frac{3\lambda}{2}}(|x| - x_0)\right]. \tag{7}$$

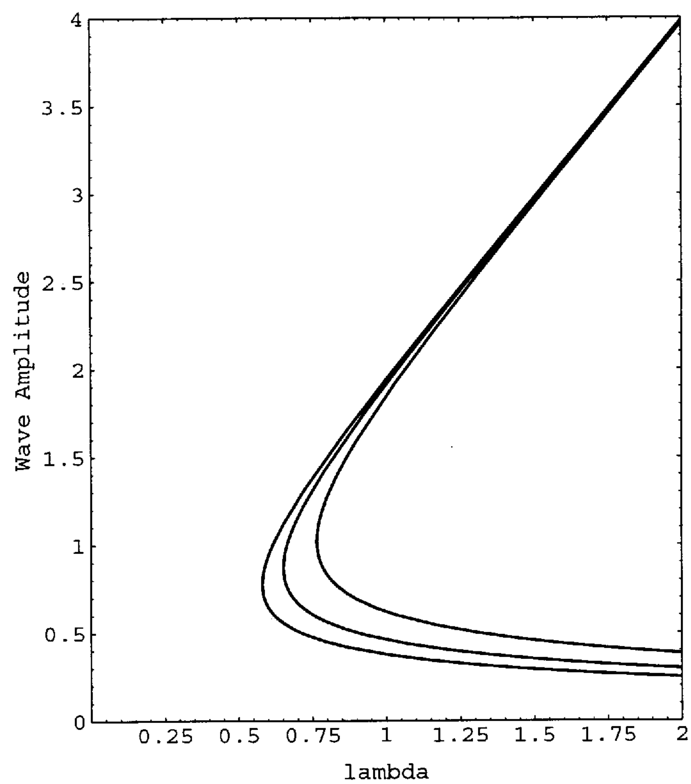

FIGURE 1. Three bifurcation diagrams of two branches of solutions of a stationary fKdV correspond to $P = 0.555360, 0.660639$ and 0.840896, respectively. The vertical coordinate stands for the dimensionless height of a solitary wave $\eta_0(0)$.

The quantity x_0 has two values and is determined by

$$f^3 - f - \frac{\sqrt{6}}{8}P\lambda^{-3/2} = 0, \quad f = \tanh\left(\sqrt{\frac{3\lambda}{2}}x_0\right). \tag{8}$$

For a given P, there are two branches of the solutions when $\lambda > \lambda_C$ as found by Malomed [**3**] and Shen [**9, 10**]. Figure 1 is the bifurcation diagram for the two solutions. The vertical coordinate stands for the height of the solitary wave and is equal to

$$\eta_0(0) = 2\lambda \text{sech}^2\left[\sqrt{\frac{3\lambda}{2}}x_0\right].$$

For $P = 0.555360$, 0.660639 and 0.840896, the corresponding λ_C values are 0.580062, 0.651230 and 0.764868, respectively. The upper branch indicates that the wave amplitude increases with respect to the Froude number. From the mathematics of solitary waves in a flat channel, this conclusion seems right but physically is incorrect. The water system can support only a solitary wave lower than $0.83322H$ when the channel is flat. When a bump is present, the height is even lower because of the bump disturbance of stationary water.

In the case of a semi-elliptical bump of height $H/4$ and base H, the value of P is 0.555360. When $\lambda = 1.0$, the Froude number is

$$F = 1 + \lambda\sqrt{\|h\|_\infty/H} = 1.5.$$

The amplitude of the upper solitary wave is $\varepsilon H\|\eta_0\|_\infty = 0.969H$, greater than $0.83322H$. This solitary wave is too high to be supported by the water. Thus, this solitary wave cannot possibly be stable. The amplitude of the lower solitary wave is $0.189H$, and this wave is stable.

As the Froude number decreases, the amplitude of the stable solitary wave increases, in contrast to what occurs in the case of flat bottom, where the amplitude of the stable traveling solitary wave increases with respect to the Froude number. For the same semi-elliptical bump of height $H/4$ and base H, when the Froude number is 1.3, the amplitude of the lower solitary wave is $0.323H$, which is about 30 percent of the depth H. This height is comparable with that of the solitary wave observed by Russell, except that his solitary wave was forced by a boat on the water surface and was traveling, whereas the solitary wave here is trapped on the bump's location and standing.

The solitary wave of maximum height is achieved when the Froude number is $F_C = 1+\lambda_C\sqrt{\|h\|_\infty/H}$. For the same semi-elliptical bump of height $H/4$ and base H, the Froude number for the maximum solitary wave is $F_C = 1.290$, and the amplitude of this maximum solitary wave is $0.386H$. While the maximum height of the solitary waves over a flat shallow channel is, of course, still $0.833H$, the maximum height of a solitary wave trapped over a bump increases with respect to the bump's area. When the semi-elliptical bump is replaced by a larger semicircular bump of radius $H/2$, the maximum height of the solitary wave is then $0.613H$, and the corresponding Froude number is $F_C = 1.460$.

Of course, all the above predictions are based upon the fKdV model, and the numerical solutions of the Euler equations should yield more accurate results. Fortunately, it has been often shown that the simple fKdV model is actually quite accurate for both the stationary case, as studied by Shen [**2, 11**], and the nonstationary case, as studied by Lee [**12**]. The relative errors among the experiments, numerical simulations from Euler equations and fKdV theory are less than 10 percent. Therefore, there is good reason to believe that when the amplitude of the maximum solitary wave is not very big, say less than $0.5H$, the predictions from the fKdV theory are highly accurate.

4. Stability of the multiple solitary wave solutions. Shen et al. [**5**] numerically demonstrated the stability of the lower solitary wave and the evolution of the higher solitary wave. This higher solitary wave degenerates into the lower solitary wave and radiates a soliton upstream and a ripple downstream. Asymptotically, the upstream soliton, the stable lower solitary wave and the downstream wake are separated from each other. Both the upstream soliton and the downstream wake propagate at a constant speed.

If a stationary solution is stable, a small perturbation will not change its profile dramatically; otherwise, it will. The stability of the stationary fKdV solitary waves (7) is defined with respect to the original time-dependent fKdV equation. Hence, we solve the initial value problem for the time dependent fKdV equation

$$\eta_{1t} + \lambda\eta_{1x} - \frac{3}{2}\eta\eta_{1x} - \frac{1}{6}\eta_{xxx} = \frac{P}{2}\delta_x(x), \tag{9}$$

$$\eta(x, t = 0) = \eta_0(x), \quad \eta^{(n)}(x = \pm\infty, t = 0), \quad n = 0, 1, 2. \tag{10}$$

This initial boundary value problem is solved by the same semi-implicit spectral method used in Shen et al. [**5**]. The method is very efficient, highly accurate and does not need an arbitrary viscosity function or a sin function correction of dispersion terms used in the conventional spectral method. In our spectral scheme, as in other spectral schemes, the equation (9) is integrated in time by the leap-frog finite difference scheme in the spectrum space. The infinite interval is replaced by $-L < x < L$ with L sufficiently large such that the periodicity assumption $u(x+L,t) = u(x-L,t) = 0$ holds, and the fast Fourier transform can be applied. The interval $(-L, L)$ is equipartitioned into N subintervals, where N is an integer power of 2. The major advantage of the spectral method is its ability to maintain the cusp in our problem even with a relatively large space mesh in x. This ability is due to the fast convergence of the Fourier series of a continuous function that has a small total variance.

Both truncation errors and round-off errors are regarded as the perturbation to the stationary solitary waves. When both the spatial mesh and temporal mesh are very small, the spectrum of the error is very wide and can be regarded as the *white* noise perturbation to the stationary solitary wave. However, as the mesh sizes increase, the power spectrum of the errors shifts toward low frequencies, and hence the error becomes *red.* According to the definition of *stability*, a profile is stable only when it is stable subject to a perturbation of any frequency. Thus, in our numerical simulation, the mesh sizes are chosen to be sufficiently small to reflect the width of the noise spectrum.

Many numerical experiments were carried out to test the stability of the lower branch of the solitary wave and the instability of the upper solitary waves for different bump shapes, sizes and various upper-stream velocities. For the initial value problem of an unforced KdV equation with zero boundary conditions at infinity, if the initial profile has a sufficiently large area under the curve, the collapse of the initial profile results in many solitons, each one smaller than the other and the largest moving the fastest. From the bifurcation diagram for the fKdV, when λ increases, the amplitude of the unstable stationary solitary wave also increases and so does the area with it. One may conjecture that, when λ is sufficiently large, the collapse of the unstable solitary wave will radiate many solitons. Again, this

assumption is incorrect. As a matter of fact, regardless of the size of λ, the collapse of the unstable solitary wave radiates only one soliton upstream. Malomed [**3**] predicted this result, but failed to provide a mathematical proof or a numerical demonstration of this prediction. We arrived at this conclusion from numerous numerical simulations. When $P = 4.0$ and $\lambda = 3.0$, the numerical simulation of the collapse of

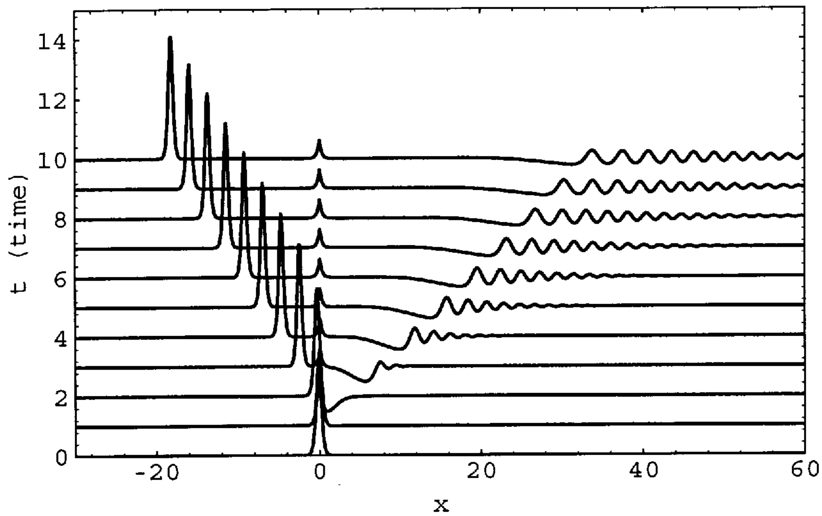

FIGURE 2. The higher solitary wave is unstable and degenerates into the stable (lower) solitary wave. The process radiates only one soliton upstream and a wake downstream propagating at a constant speed.

the higher solitary wave is shown in Figure 2, where the wave profile is scaled down by a factor of 0.4 for the purpose of clear display.

Our numerical simulations appeared to imply that the amplitude of the upstream radiated soliton is linearly proportional to λ. When $P = 0.555360$, the relationship of the amplitude versus λ is

$$a_s = 3.24\lambda. \tag{11}$$

When the usptream radiated soliton is far away from the forcing site, its motion is governed by the KdV without forcing. The amplitude a_s and the speed s of the soliton satisfy the relationship

$$a_s = 2(\lambda + s). \tag{12}$$

It would be desirable to have an analytical formula that predicts either a_s or s from λ for a given bump. However, such a formula appears to be difficult to find.

5. Conclusions and discussion. Based upon the maximum height of solitary waves being $0.83322H$, we have given a simple explanation of the stability and instability of solitary waves in a forced KdV equation. We have concluded that: (i) the higher branch of the solitary waves is unstable since they can be higher than the maximum height of solitary waves in a channel; (ii) the lower branch of solitary waves are stable, as was numerically justified by Shen et al. [**5**]; (iii) for a given bump, the height of the stable solitary wave decreases as the Froude number increases; and (iv) the unstable higher solitary wave degenerates into the lower and stable solitary wave and radiates only one soliton upstream and a ripple downstream that travels at a constant speed.

As pointed out earlier, the upstream-running single soliton is not a stable water-surface profile. The KdV model fails to provide a good approximation in this region for this particular case. One must use the full Euler equation to model upstream water surface. This is not contradictory because almost all the asymptotic solutions of a partial differential equation problem need justification of validity region in terms of parameters and space-time independent variables.

If the soliton is too high to be supported by water, perhaps the soliton mass will distribute in water ripples. If this is the case, then the higher unstable fKdV soliar wave will degenerate into the lower, stable solitary wave and radiate ripples both downstream and upstream.

There are other mathematical studies on the stability of fKdV solitary waves, such as that of Camass and Wu [**13**]. They considered various cases: nonlinear stability, linear stability and oscillation. Our work is different from this mathematical study because we wish to provide a simple stability criterion and try to demonstrate what will happen after the instability. It is our understanding that (i) the existence of the higher solitary wave in an Euler system still needs a rigorous mathematical proof, (ii) a rigorous proof of the stability of the lower solitary wave is still not available and (iii) the instability process of the higher solitary wave still requires further investigation using the full Euler equations.

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