The Product of Chord Lengths of a Circle

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As anyone who has ever had any exposure to complex variables is aware, the subject is full of surprises. From applications in fluid dynamics to the closed-form summation of infinite series, complex analysis has applications that would seem to have no relation to a theory concerned with the "imaginary" number $\sqrt{-1}$. This paper presents an intriguing result in geometry that can be derived by setting the problem in the complex plane and applying the theory of residues (see also [2], p. 69, problem 44).

Suppose we have a circle of unit radius, whose circumference is divided into 8 equal arcs by 8 points. Suppose also that we draw lines from one of the points to each of the other seven points as shown in FIGURE 1. Consider then the problem of determining the product of the 7 chord lengths. It turns out that this product is equal to the number of points we started with, namely 8. In fact, as we shall show below, if we start with n points, the product of the $n-1$ chord lengths we construct will *always* be equal to n. One can easily check this result for the cases $n = 2$, 3, and 4.

For the general result, suppose we have a circle of unit radius and n points that divide the circumference into n equal arcs. Let $c_1, c_2, \ldots, c_{n-1}$ denote chords drawn from one of the points to each of the remaining $n-1$ points (see FIGURE 1). The product of the $n-1$ chord lengths is just n, i.e.

$$\prod_{k=1}^{n-1} |c_k| = n, \tag{1}$$

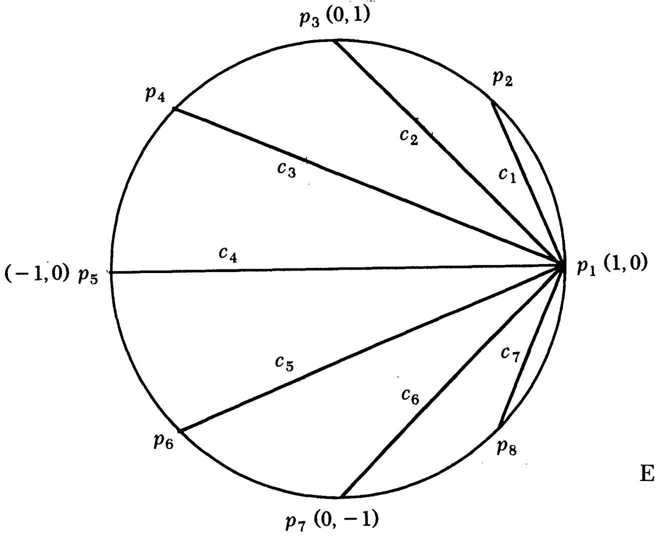

FIGURE 1
Example: $n = 8$.

where $|c_k|$ denotes the length of the chord $c_k, k = 1, \ldots, n-1$ (see also **[3]**, p. 32, problem 160 and **[1]**, pp. 33–34, problems 4.19, 4.20 for related results).

Proof. Without loss of generality, let the originating point be the point $(1,0)$ in the complex plane. Then the n points can be represented in the complex plane by

$$p_k = e^{i\frac{2\pi(k-1)}{n}}.$$

Since chord c_k is the line from p_1 to p_{k+1}, we have

$$\prod_{k=1}^{n-1} |c_k| = \prod_{k=1}^{n-1} \left|1 - e^{i\frac{2\pi k}{n}}\right|. \tag{2}$$

Consider the function

$$f(z) = \frac{1}{z^n - 1} = \frac{1}{\prod_{k=1}^{n} \left(z - e^{i\frac{2\pi k}{n}}\right)}.$$

The calculation of the residue of f at $z = 1$ can be done by using the formula

$$\operatorname{Res}(f,1) = \lim_{z \to 1} (z-1) f(z) = \frac{1}{\prod_{k=1}^{n-1} \left(1 - e^{i\frac{2\pi k}{n}}\right)}. \tag{3}$$

Since f has a simple pole at $z = 1$, $\operatorname{Res}(f, 1)$ can also be calculated by

$$\operatorname{Res}(f,1) = \frac{1}{\frac{d}{dz}(z^n - 1)\Big|_{z=1}} = \frac{1}{n}. \tag{4}$$

Hence $\prod_{k=1}^{n-1} |c_k| = n$.

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